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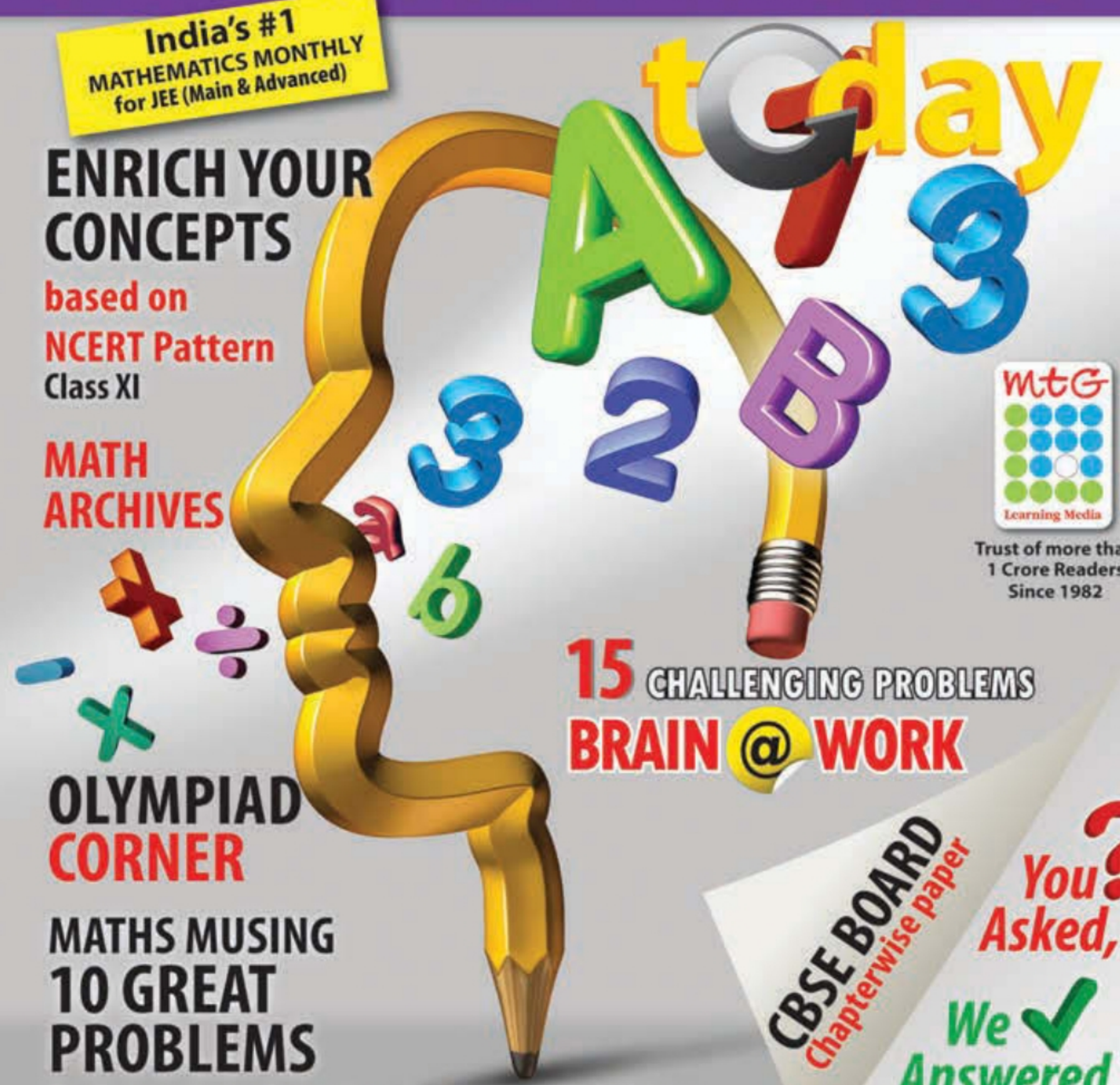

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editorial

Men Behind India's Success Stories

There is a general feeling that if only all our scientists from abroad come back to India to work, we shall be much more advanced in science and technology. This is a fallacy. From the vedic times, Indians had been in the avant-garde of Mathematics, metallurgy and science. Science in India is what it is today because of the dedicated work of our scientists, in mathematics, physical and natural sciences, medicine and so on.

We should seek knowledge from all over the world, but we should develop our own knowledge. Now-a-days there are ratings given for universities. We do not know what is the basis of this rating. Madras University was respected outside because of Ramanujan whose abilities were recognized by a British mathematician. I do not know what is the rating of the Kerala Engineering College or Coimbatore Engineering College in International ratings. All that we know is the students of these universities have set a standard that will be very difficult for the alumini of the universities elsewhere to achieve not to speak of surpassing.

Our scientists have given the courage to develop on our own. The calculations of the scientists are so perfect that the very first attempt was a success. Call the scientists what you want – physicists, chemists or mathematicians. We are proud of our scientists. We call on our students to develop their faculties in every branch. We should proudly say – we had our training in India and we shall keep our flag up. We have chosen this topic for Maths, because the success of our Mission to Mars is equally due to perfect calculations of our scientists.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 143

JEE MAIN

1. If $\omega^3 = 1$, $\omega \neq 1$ and a, b, c are real such that

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega} = 2\omega^2,$$

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2} = 2\omega,$$

then $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} =$

(a) 1 (b) –1 (c) 2 (d) –2

2. $\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{6}\right)\sec\left(x-\frac{\pi}{3}\right)dx =$

(a) $\ln\sqrt{3}$ (b) $\ln 3$
(c) $2\ln 3$ (d) $3\ln 3$

3. The area bounded by the parabola $x^2 = 4y$ and its normal through the point (1, 2) is

(a) 20 sq. units (b) 21 sq. units
(c) $\frac{61}{3}$ sq. units (d) $\frac{64}{3}$ sq. units

4. The distance of the point (3, 0, 5) from the line $x - 2y + 2z - 4 = 0 = x + 3z - 11$ is

(a) 1 (b) 2 (c) 3 (d) 4

5. In a triangle ABC, $C = \frac{\pi}{2}$ and D is a point on the side CB. The circle with centre D and radius 1 touches sides AC and AB. If $s = 4$, then $\Delta =$

(a) $\frac{5}{3}$ (b) $\frac{7}{3}$ (c) $\frac{8}{3}$ (d) $\frac{10}{3}$

JEE ADVANCED

6. The number of subsets $\{x, y\}$ of the set of the first 50 natural numbers such that $x^2 - y^2$ is divisible by 7, is divisible by

(a) 2 (b) 5 (c) 7 (d) 11

COMPREHENSION

If $z - \frac{1}{z} = i$, then

7. $z^{2015} - \frac{1}{z^{2015}} =$

(a) i (b) $-i$ (c) ± 1 (d) $\pm\sqrt{3}$

8. $z^{2015} + \frac{1}{z^{2015}} =$

(a) i (b) $-i$ (c) ± 1 (d) $\pm\sqrt{3}$

INTEGER MATCH

9. Four dice are rolled once, the probability of getting the sum of the numbers on the dice, is atleast 10, is $\frac{m}{n}$, reduced fraction where $(n - m)$ is

MATCHING LIST

10.

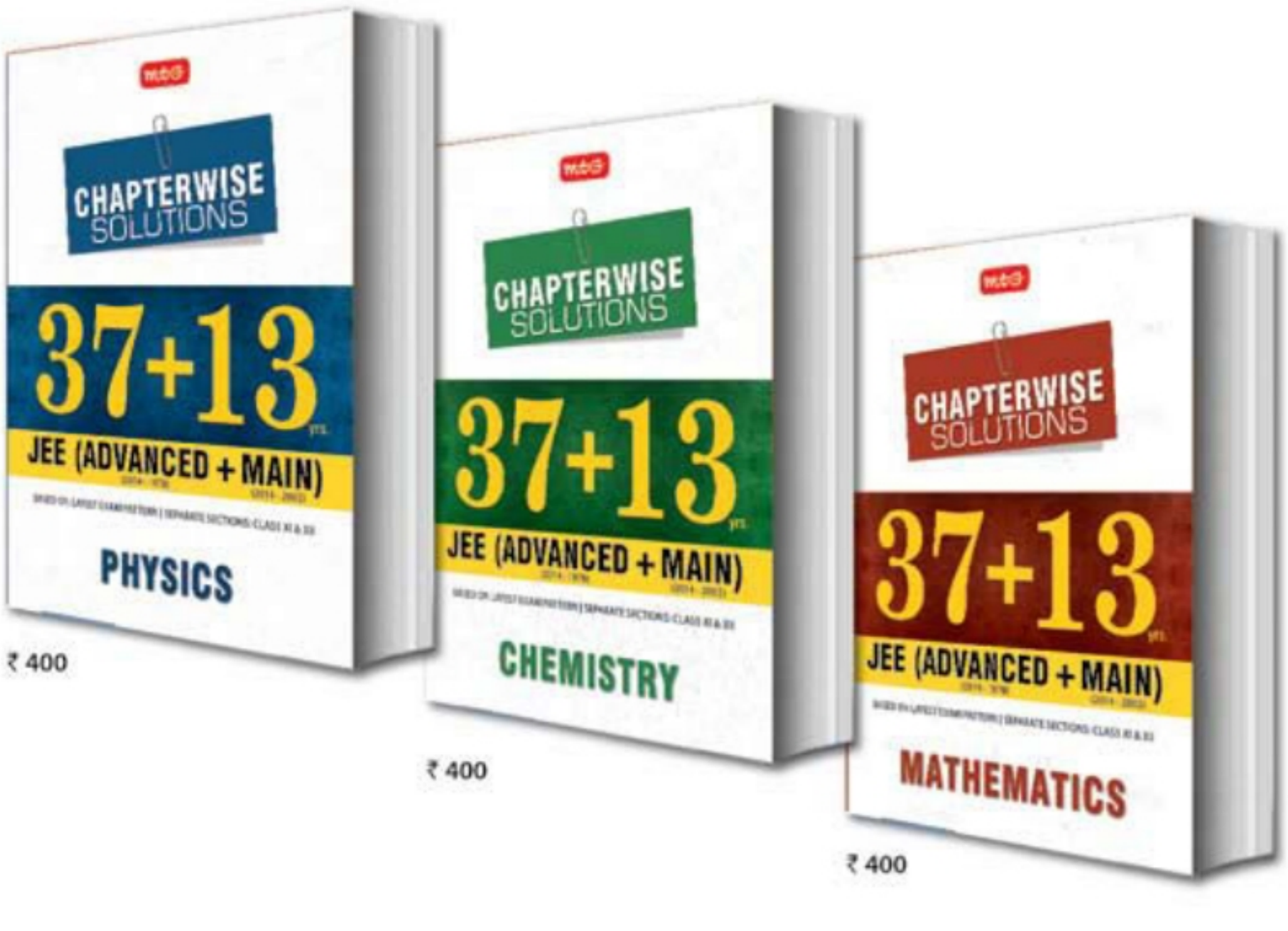
	Column-I		Column-II
P.	In a triangle ABC, a, b, c are in A.P., the maximum value of B is	1.	$\frac{\pi}{4}$
Q.	In a triangle ABC, r_1, R, r_2 are in A.P., then C is	2.	$\frac{2\pi}{3}$
R.	If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = \cos\theta$, then the maximum value of θ is	3.	$\frac{\pi}{2}$
S.	$\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}(2r^2)$ is	4.	$\frac{\pi}{3}$

	P	Q	R	S
(a)	2	1	4	3
(b)	1	4	3	2
(c)	4	3	2	1
(d)	3	2	1	4

See Solution set of Maths Musing 142 on page no. 72.

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

Mad about rehearsing?



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MOCK TEST PAPER 2015

JEE Main & Advanced

Series-5

Differential Calculus : Functions, composition of two functions and inverse of a function, limits, continuity, derivative, chain rule, derivatives of implicit functions and derivative of the functions defined parametrically. L'Hospital's rule (statement only) and applications. Second order derivative.

Probability : Classical definition, addition rule, conditional probability and Baye's theorem, independent events, multiplication rule.

PAPER-I

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. A problem in mathematics is given to three students A, B and C and their respective probability of solving the problem is $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. Then the probability that the problem is solved, is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{7}{8}$

2. The probability that at least one of the events A and B occur is $\frac{3}{5}$. If A and B occur simultaneously with probability $\frac{1}{5}$, then the value of $P(A') + P(B')$ is

(a) $\frac{2}{3}$ (b) $\frac{4}{5}$ (c) $\frac{6}{5}$ (d) $\frac{7}{5}$

3. The probability that a regular scheduled flight departs on time is 0.9, the probability that it arrives on time is 0.8 and the probability that it departs and arrives on time is 0.7. Then the probability that a plane arrives on time, given that it departs on time, is

(a) 0.72 (b) $\frac{8}{9}$ (c) $\frac{7}{9}$ (d) 0.56

4. In a quadratic equation $ax^2 + bx + c = 0$, the coefficients a, b, c take value from the set $\{1, 2, 3\}$. The probability that roots of the equation are real is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{4}{27}$

5. In a bolt factory three machines A, B and C manufacture 2000, 2500 and 4000 bolts respectively every day. Of their outputs 3%, 4% and 2.5% are defective bolts. One bolt is drawn at random from a day's production and is found to be defective. Then the probability that it was produced by machine C is

(a) $\frac{3}{13}$ (b) $\frac{4}{13}$ (c) $\frac{5}{13}$ (d) $\frac{7}{13}$

6. The inverse of the function $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is

(a) $f^{-1}(x) = \frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$

(b) $f^{-1}(x) = \log_{10}(2 - x)$

(c) $f^{-1}(x) = \frac{1}{2}\log_{10}(2x - 1)$

(d) $f^{-1}(x) = \frac{1}{4}\log_{10}\left(\frac{2x}{2-x}\right)$

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397

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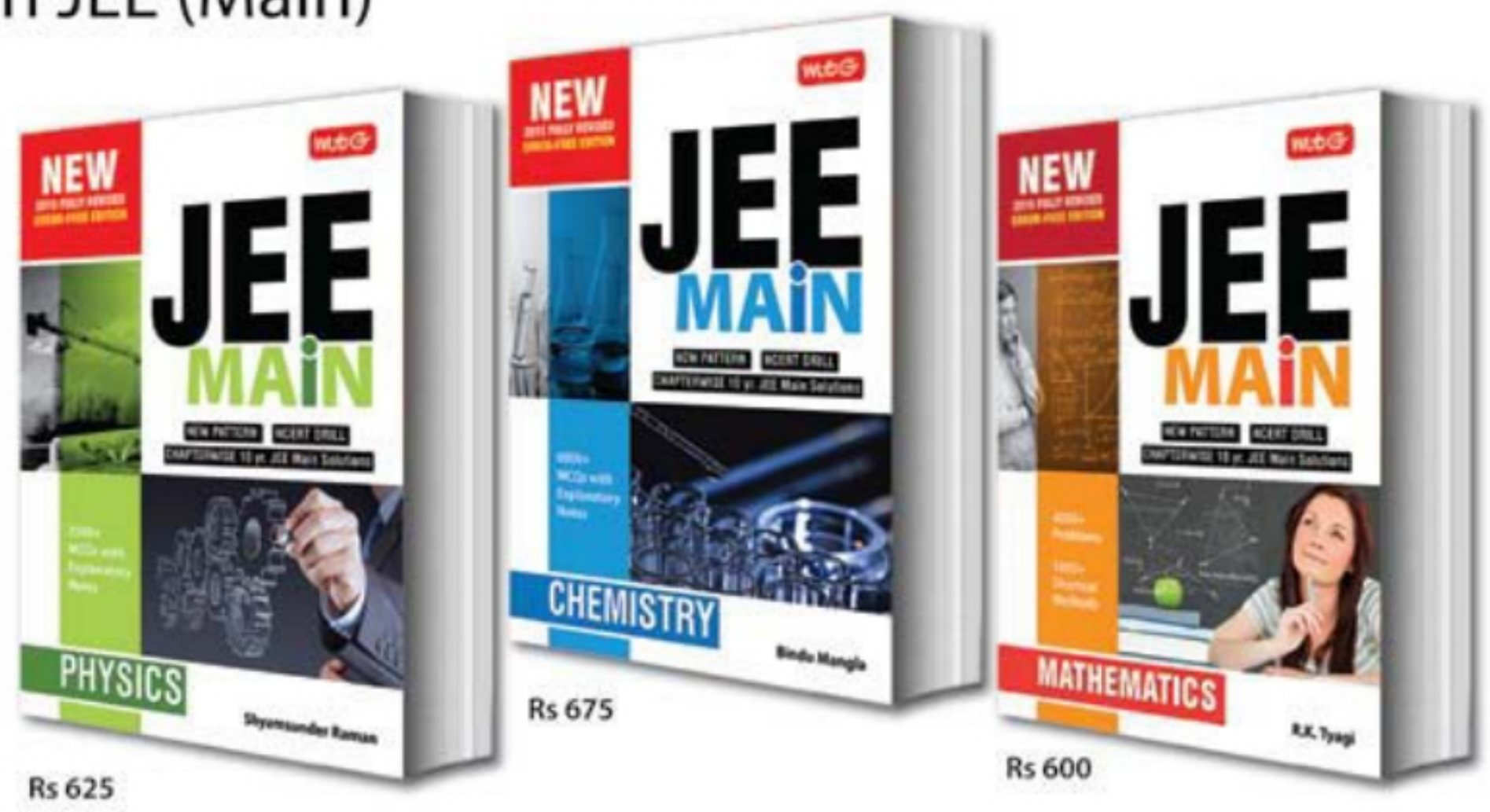

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7. The domain of the function

$$f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x) \text{ is}$$

(a) $(1, 2)\cup(2, \infty)$
(b) $(-1, 0)\cup(1, 2)\cup(2, \infty)$
(c) $(1, 2)$
(d) $(-1, 0)\cup(1, 2)$

8. The value of $\lim\limits_{h\to 0}\dfrac{e^{(x+h)^2}-e^{x^2}}{h}$ is

(a) xe^{x^2} (b) $2e^{x^2}$ (c) $4xe^{x^2}$ (d) $2xe^{x^2}$

9. If $f(x)=\begin{cases} x & , \text{ when } 0\le x\le 1 \\ 2x-1 & , \text{ when } x>1 \end{cases}$ then

(a) $f(x)$ is continuous but not differentiable at $x = 1$
(b) $f(x)$ is discontinuous at $x = 1$
(c) $f(x)$ is differentiable at $x = 1$
(d) none of these

10. Let $f(x)$ be a differentiable even function; consider the following statements
1 : $f'(x)$ is an even function
2 : $f'(x)$ is an odd function
3 : $f'(x)$ may be even or odd.
Which of the above statements is/are correct?
(a) 1 only (b) 2 and 3
(c) 2 only (d) 1 and 3

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/ are correct.

11. Let $f(x)=\begin{cases} 0 & , \text{for } x=0 \\ x^2\sin\dfrac{\pi}{x} & , \text{for } -1<x<1,\ x\neq 0 \\ x|x| & , \text{for } x\ge 1 \text{ or } x\le -1 \end{cases}$, then

(a) $f(x)$ is an odd function
(b) $f(x)$ is an even function
(c) $f(x)$ is neither odd nor even
(d) $f'(x)$ is an even function

12. If A and B are two independent events such that

$$P(A\cap B)=\frac{1}{12} \text{ and } P(\text{either } A \text{ or } B)=\frac{1}{2}, \text{ then}$$

(a) $P(A\cup B)=\dfrac{1}{2}$ (b) $P(A^c\cap B^c)=\dfrac{1}{2}$
(c) $P(A)=\dfrac{1}{3},\ P(B)=\dfrac{1}{4}$
(d) $P(A)=\dfrac{1}{4},\ P(B)=\dfrac{1}{3}$

13. If $f(x)=x\dfrac{e^{[x]+|x|}-2}{[x]+|x|}$ (where $[x]$ is the greatest integer function), then

(a) $\lim\limits_{x\to 0^+} f(x)=-1$
(b) $\lim\limits_{x\to 0^-} f(x)=0$
(c) $\lim\limits_{x\to 0^+} f(x)$ does not exist
(d) $\lim\limits_{x\to 0} f(x)$ does not exist

14. If $f(x)=\begin{cases} \dfrac{1}{5}(2x^2+3), & \text{for } -\infty<x\le 1 \\ 6-5x, & \text{for } 1<x<3 \\ x-3, & \text{for } 3\le x<\infty \end{cases}$, then f is

(a) continuous at $x = 1$
(b) discontinuous at $x = 1$
(c) continuous at $x = 3$
(d) none of these

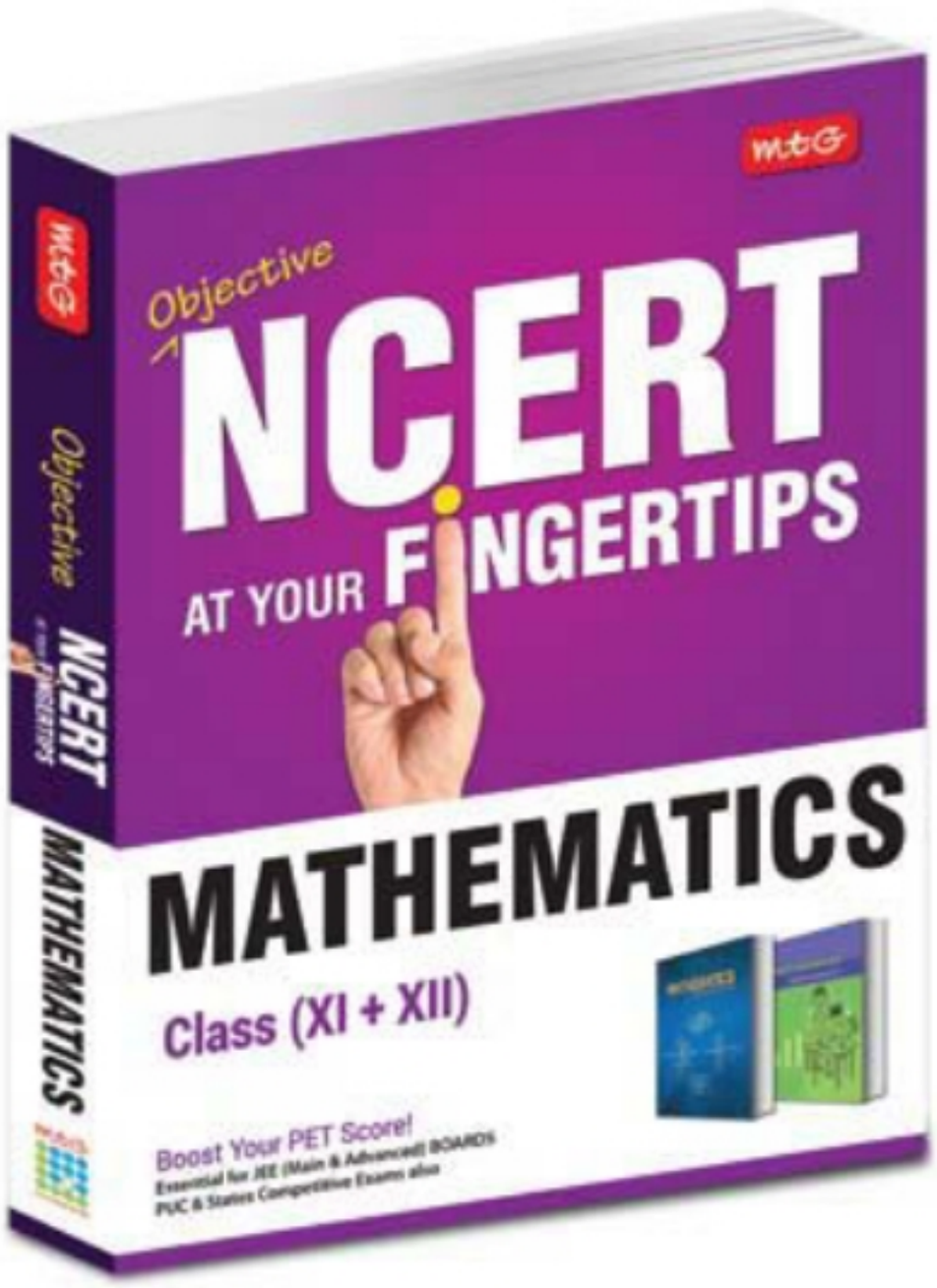
15. Let $f(x) = \min\{x, x^2\}$ (where $\{x\} = x - [x]$ and $[x]$ denotes the greatest integer function), then
(a) f is continuous for all $x\in R$
(b) f is differentiable for all $x\in R$
(c) f is differentiable for all $x\in R$ except 0, 1
(d) none of these

SECTION-III

Assertion & Reason Type

In each of the following questions, two statements are given Statement-1 (Assertion) and Statement-2 (Reason). Examine the statements carefully and answer the questions according to the instruction given below:
(a) If both A and R is correct and R is a proper reason of A.
(b) If both A and R is correct and R is not the proper reason of A.
(c) If A is correct and R is wrong.
(d) If A is wrong and R is correct.

How to choose the right answer, fast?



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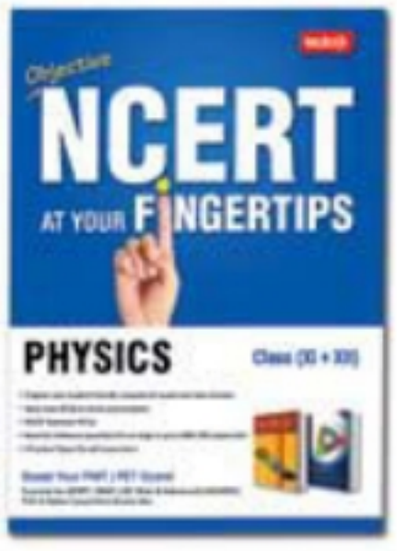

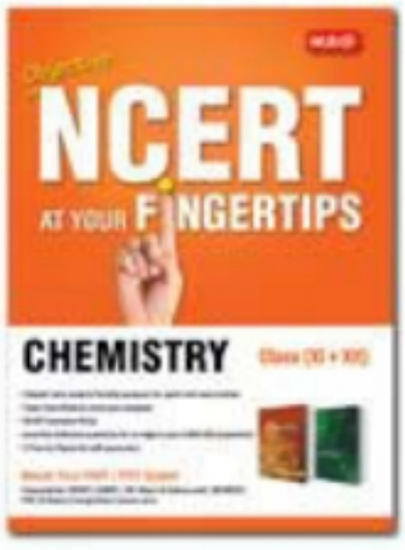

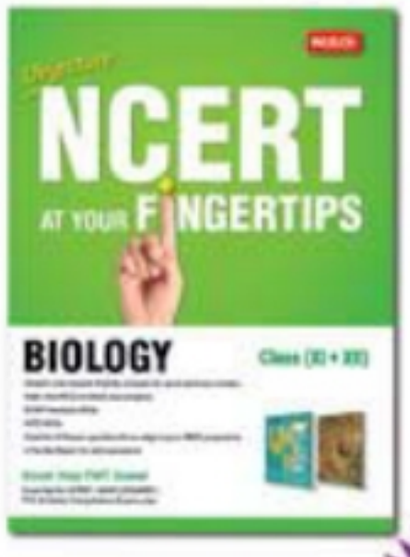

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16. **Statement-1 :** If $\frac{1}{5}(1+5p), \frac{1}{3}(1+2p), \frac{1}{3}(1-p)$ and $\frac{1}{5}(1-3p)$ are probabilities of four mutually exclusive events, then p can take infinite number of values.
Statement-2 : If A, B, C and D are four mutually exclusive events, then
$P(A), P(B), P(C), P(D) \geq 0$
and $P(A) + P(B) + P(C) + P(D) \leq 1$

17. **Statement-1 :** Let $A = \{2, 3, 7, 9\}$ and $B = \{4, 9, 49, 81\}$, $f: A \to B$ is a function defined as $f(x) = x^2$. Then f is a bijection from A to B.
Statement-2 : A function f from a set A to a set B is a bijection if $f(A) = B$ and $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ for all $x_1, x_2 \in A$ and $n(A) = n(B)$.

18. **Statement-1 :** If $\lim\limits_{x\to 0}\left(\frac{\sin x}{x} + f(x)\right)$ does not exist, then $\lim\limits_{x\to 0} f(x)$ does not exist.
Statement-2 : $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

19. Let $f: R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$
Statement-1 : $f(c) = \frac{1}{3}$ for some $c \in R$.
Statement-2 : $0 < f(x) \leq \frac{1}{2\sqrt{2}}$ for all $x \in R$.

20. Four numbers are chosen at random (without replacement) from the set {1, 2, 3,....., 20}
Statement-1 : The probability that chosen numbers when arranged in some order will form an A.P. is $\frac{1}{85}$.
Statement-2 : If the four chosen numbers form an A.P., then the set of all possible values of common difference is {±1, ±2, ±3, ±4, ±5}

SECTION-IV

Integer Answer Type

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like as given.

X	Y	Z	W
0	**0**	0	0
1	1	1	1
2	2	2	**2**
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	**9**	9

21. There are four letters and four envelopes bearing addresses at random. The probability that the letters are placed in correct envelopes is $\frac{K}{24}$, then the value of K is

22. If A and B are two independent events and $P(A) = \frac{3}{8}, P(B) = \frac{1}{2}, P(A \cap B) = \frac{1}{4}$, $P(A \cup B)^c = \frac{K}{8}$, then find the value of K.

23. Find the value of $f(0)$ for which
$f(x) = \frac{64(\sqrt{x+4} - 2)}{\sin 2x}$ is continuous.

24. Suppose that f is differentiable for all x and that $f'(x) \leq 2\ \forall\ x$. If $f(1) = 2$ and $f(4) = 8$, then $f(2) =$

25. The value of $\lim\limits_{x\to\frac{\pi}{4}}\frac{8 - \sqrt{2}(\cos x + \sin x)^5}{1 - \sin 2x} = 5\lambda$; then the value of λ is

PAPER-II

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. Solve the equation $4\{x\} = x + [x]$, where $\{x\}$ and $[x]$ are respectively the fractional part function and greatest integer function.

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

2. The graph of the function $y = x^2 - px + x + 9$ is always above the x-axis if

(a) $p > 7$ (b) $p > -9$
(c) $-5 < p < 7$ (d) $p < -5$ or $p > 7$

3. The domain of the function $f(x)=\sqrt{x(2-x)}\cdot\log(x-1)$ is

(a) $1<x<2$ (b) $0\le x\le 2$

(c) $1<x\le 2$ (d) $0<x<2$

4. If $f(x)$ is an even function and satisfies the relation $x^2 f(x)-2f\left(\frac{1}{x}\right)=g(x)$, where $g(x)$ is an odd function, then $f(10)$ equals

(a) 0 (b) $\frac{50}{75}$ (c) $\frac{49}{79}$ (d) 1

5. The period of $f(x)=\sin^4 x+\cos^4 x$ is

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $\frac{3\pi}{2}$

6. $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(a) exist and its value is $\sqrt{2}$

(b) exist and its value is $-\sqrt{2}$

(c) does not exist as $x-1\to 0$

(d) does not exist as L.H.L is not equal to R.H.L

7. The value of $\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ is

(a) 2 (b) –2 (c) 1/2 (d) –1/2

8. One Indian and four American men and their wives are to be seated randomly around a circular table. Then the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

9. A signal which can be green or red with respective probability $\frac{4}{5}$ and $\frac{1}{5}$ is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is

(a) 3 (b) $\frac{6}{5}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

10. Five horses are in a race. A man selects two of them at random and bets on them. All the horses have equal chances of winning. The probability that the man selected the winning horse is

(a) $\frac{3}{5}$ (b) $\frac{1}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{25}$

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

11. Let $f(x)=\begin{cases}\{1+|\sin x|\}^{\frac{a}{\sin x}} & ,-\frac{\pi}{6}<x<0\\ b & ,x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & ,0<x<\frac{\pi}{6}\end{cases}$

is continuous at $x=0$. Then the value of

(a) $a=\frac{2}{3}$ (b) $[a]=0$

(c) $b=e^{2/3}$ (d) $[a]=1$

[.] is greatest integer function

12. Let $f(x)=\begin{cases}2x+3, & \text{when } x\le 1\\ ax^2+bx, & \text{when } x>1\end{cases}$

is differentiable everywhere, then

(a) $a=-3$ (b) $b=8$

(c) $f'(x)=-6x+8$ (d) $f'(2)=-4$

13. If $f(x)=\cos\left[\frac{\pi^2}{2}\right]x+\sin\left[-\frac{\pi^2}{2}\right]x$, [.] denotes the greatest integer function, then

(a) $f(0)=1$ (b) $f\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}+1}$

(c) $f\left(\frac{\pi}{2}\right)=0$ (d) $f(\pi)=0$

14. The probabilities that a student passes in mathematics, physics and chemistry are m, p and c respectively. Of these subjects, a student has a 75% chance of passing in at least one, a 50% chance of passing in at least two, and a 40% chance of passing in exactly two subjects. Which of the following relations are true?

(a) $p+m+c=\frac{19}{20}$ (b) $p+m+c=\frac{27}{20}$

(c) $pmc=\frac{1}{10}$ (d) $pmc=\frac{1}{4}$

15. If A and B are two independent events such that $0 < P(A) < 1, 0 < P(B) < 1$, then

(a) A, B are mutually exclusive

(b) A and B^c are independent

(c) A^c and B^c are independent

(d) $P(A/B) + P(A^c/B) = 1$

SECTION-III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

Paragraph for Question No. 16 to 18

A fair die is tossed repeatedly until a six is obtained. Let X denotes the number of tosses required.

16. The probability that $X = 3$ equals

(a) $\frac{25}{216}$ (b) $\frac{25}{36}$ (c) $\frac{5}{36}$ (d) $\frac{125}{216}$

17. The probability that $X \geq 3$ equals

(a) $\frac{125}{216}$ (b) $\frac{25}{216}$ (c) $\frac{5}{36}$ (d) $\frac{25}{36}$

18. The conditional probability that $X \geq 6$ given $X > 3$ equals

(a) $\frac{125}{216}$ (b) $\frac{25}{216}$ (c) $\frac{5}{36}$ (d) $\frac{25}{36}$

Paragraph for Question No. 19 to 21

Let $f(x)$ be a real valued function such that

$(x^2 - y^2)f(x)f(y) = xy((x-y)f(x+y) + (x+y)f(x-y))$

$\forall\ x, y \in R$ and $f'(0) = 2$

19. $f(x)$ will be

(a) an even function (b) an odd function

(c) can't say (d) neither even nor odd

20. $f(\ln 2)$ is

(a) $\frac{5}{3}\ln 3$ (b) $\frac{5}{4}\ln 4$ (c) $\frac{5}{3}\ln 2$ (d) $\ln 4$

21. $\lim_{x\to 0}\left[\frac{f(x)}{\sin x}\right]$ is equal to (where [.] denote greatest integer function)

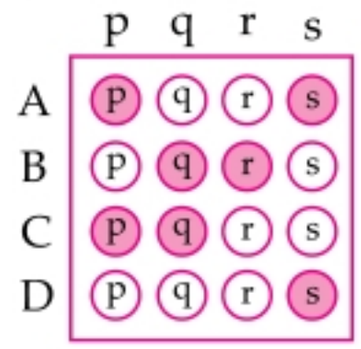
(a) 1 (b) 2

(c) 3 (d) does not exist

SECTION-IV

MATRIX-MATCH TYPE

This section contains 3 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s, then the correct darkening of bubbles will look like the above.



22. Match the following : Let $f(x) = \lim_{x\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$

Column I		Column II	
(A)	$f(x) = 1$ on	(p)	$(1, \infty)$
(B)	$f(x) = -1$ on	(q)	$(2, \infty)$
(C)	$f(x) \geq 0$ on	(r)	$[2, \infty)$
(D)	$f(x) = \text{sgn}(\lvert x\rvert - 1)$	(s)	$\left(-\frac{1}{2}, \frac{1}{2}\right)$

23. Match the following :

Column I		Column II	
(A)	$f(x) = \lvert x^3\rvert$ is	(p)	continuous in $(-1, 1)$
(B)	$f(x) = \sqrt{\lvert x\rvert}$ is	(q)	differentiable in $(-1, 1)$
(C)	$f(x) = \lvert\sin^{-1}x\rvert$	(r)	differentiable in $(0, 1)$
(D)	$f(x) = \cos^{-1}\lvert x\rvert$ is	(s)	not differentiable at least at one point $(-1, 1)$

SOLUTIONS

PAPER-I

1. (a) : Given that

$$P(A)=\frac{1}{2}, P(B)=\frac{1}{3} \text{ and } P(C)=\frac{1}{4}$$

Now, the problem will be solved, if at least one of them can solve the problem

$\therefore\ P(A \cup B \cup C) = 1 - P(A \cup B \cup C)^c$

$= 1 - P(A^c \cap B^c \cap C^c)$

$= 1 - P(A^c)P(B^c)P(C^c)$

$$=1-\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)$$
$$=1-\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=1-\frac{1}{4}=\frac{3}{4}$$

2. (c): $P(A\cup B)=\frac{3}{5}$ and $P(A\cap B)=\frac{1}{5}$

$\therefore\quad P(A)+P(B)-P(A\cap B)=\frac{3}{5}$

$\Rightarrow\quad P(A)+P(B)-\frac{1}{5}=\frac{3}{5}\quad\Rightarrow\quad P(A)+P(B)=\frac{4}{5}$

$\Rightarrow\quad 1-P(A^c)+1-P(B^c)=\frac{4}{5}$

$\Rightarrow\quad P(A^c)+P(B^c)=2-\frac{4}{5}=\frac{6}{5}$

3. (c): Let E_1 and E_2 be the respective events that the plane departs and arrive on time.

$\therefore$ We have $P(E_1)=0.9$, $P(E_2)=0.8$

and $P(E_1\cap E_2)=0.7$

$\therefore\quad P(E_2/E_1)=\frac{P(E_1\cap E_2)}{P(E_1)}=\frac{0.7}{0.9}=\frac{7}{9}$

4. (d) : Here it is clear that sample space consist of $3^3=27$ points, for the event A and the roots of the quadratic equation are real.

Roots of the equation will be real if $b^2-4ac\geq 0$ *i.e.* $b^2\geq 4ac$

Favourable cases are as follows :

a	c	ac	b (where $b^2\geq 4ac$)	No. of cases
1	1	1	2, 3	2
1	2	2	3	1
2	1	2	3	1
1	3	3	–	–

Thus the total number of favourable cases are 4

$\therefore\quad P(A)=\frac{4}{27}$

5. (c) **6. (a)**

7. (b) : $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$

$f(x)$ will be defined when $4-x^2\neq 0$

$\Rightarrow\quad x^2\neq 4\quad\Rightarrow\quad x\neq\pm 2$ ...(i)

and $x^3-x>0\quad\Rightarrow\quad x(x^2-1)>0$

$\Rightarrow\quad x(x+1)(x-1)>0$

i.e. $x>0$ and $x>1$ ...(ii)

or $x<0$ and $-1<x<0$ ...(iii)

From (i), (ii) and (iii), we get

$x\in(-1,0)\cup(1,2)\cup(2,\infty)$

8. (d) : $\underset{h\to 0}{\text{Lt}}\ \frac{e^{(x+h)^2}-e^{x^2}}{h}$

$=\underset{h\to 0}{\text{Lt}}\ \frac{e^{x^2+2xh+h^2}-e^{x^2}}{h}$

$=e^{x^2}\underset{h\to 0}{\text{Lt}}\left(\frac{e^{h(2x+h)}-1}{h(2x+h)}\right)(2x+h)$

$=e^{x^2}(2x+0)=2xe^{x^2}$

9. (a) : $f(x)=\begin{cases}x & ,\text{when}\quad 0\leq x\leq 1\\ 2x-1 & ,\text{when}\quad x>1\end{cases}$

$\underset{x\to 1^+}{\text{Lt}}\ f(x)=\underset{x\to 1}{\text{Lt}}\ 2x-1=2\cdot 1-1=1$

$\underset{x\to 1^-}{\text{Lt}}\ f(x)=\underset{x\to 1}{\text{Lt}}\ x=1$, and $f(1)=1$

Thus $f(x)$ is continuous at $x=1$

Now $Rf'(1)=\underset{h\to 0^+}{\text{Lt}}\ \frac{f(1+h)-1}{h}$

$=\underset{h\to 0}{\text{Lt}}\ \frac{\{2(1+h)-1\}-1}{h}$ [$\because f(x)=2x-1$, when $x>1$]

$=\underset{h\to 0}{\text{Lt}}\ \frac{2h}{h}\quad\Rightarrow\quad Rf'(1)=2$

$Lf'(1)=\underset{h\to 0^-}{\text{Lt}}\ \frac{f(1+h)-1}{h}$

$=\underset{h\to 0}{\text{Lt}}\ \frac{1+h-1}{h}=1$ [$\because\quad f(x)=x$ when $x<1$]

$\because\quad Rf'(1)\neq Lf'(1)$

$\therefore\quad f(x)$ is not differentiable at $x=1$

So, $f(x)$ is continuous at $x=1$, but not differentiable at $x=1$.

10. (c): Let $f(x)=\phi(x)+\phi(-x)$

$\therefore\quad f(-x)=\phi(-x)+\phi(x)$

$\Rightarrow\quad f(-x)=f(x)$

$\therefore\quad f(x)$ is an even function.

Now $f'(x)=\phi'(x)-\phi'(-x)$

Also $f'(-x)=\phi'(-x)-\phi'(x)$

$=-[\phi'(x)-\phi'(-x)]$

$\therefore\quad f'(-x)=-f'(x)$

11. (a, d)

12. (a, b, c, d) : Given that A and B are two independent events.

Also $P(A\cap B)=\frac{1}{2}$ and $P(\text{either } A \text{ or } B)=\frac{1}{2}$

$$\Rightarrow \quad P(A \cup B) = \frac{1}{2}$$

Now $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Again $P(A \cap B) = \frac{1}{12} = \frac{1}{3} \cdot \frac{1}{4} = P(A)P(B)$

$\Rightarrow$ either $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}$

or $P(A) = \frac{1}{4}, P(B) = \frac{1}{3}$

13. (a, b, d)

14. (a) : $f(x) = \begin{cases} \frac{1}{5}(2x^2+3) & , \text{for} \quad -\infty < x \le 1 \\ 6-5x & , \text{for} \quad 1 < x < 3 \\ x-3 & , \text{for} \quad 3 \le x < \infty \end{cases}$

Now, $\underset{x \to 1^+}{\text{Lt}} f(x) = \underset{x \to 1}{\text{Lt}} 6-5x = 6-5 \cdot 1 = 1$

$\underset{x \to 1^-}{\text{Lt}} f(x) = \underset{x \to 1}{\text{Lt}} \frac{1}{5}(2x^2+3) = 1$

and $f(1) = \frac{1}{5}(2 \cdot 1^2 + 3) = 1$

$\therefore$ $f(x)$ is continuous at $x = 1$

Again $\underset{x \to 3^+}{\text{Lt}} f(x) = \underset{x \to 3}{\text{Lt}} x-3 = 3-3 = 0$

$\underset{x \to 3^-}{\text{Lt}} f(x) = \underset{x \to 3}{\text{Lt}} (6-5x) = -9$

$\because$ $\underset{x \to 3^+}{\text{Lt}} f(x) \neq \underset{x \to 3^-}{\text{Lt}} f(x)$

$\therefore$ $f(x)$ is not continuous at $x = 3$.

15. (a, c)

16. (a) : Since $\frac{1}{5}(1+5p), \frac{1}{3}(1+2p), \frac{1}{3}(1-p)$ and $\frac{1}{5}(1-3p)$ are probabilities of four mutually exclusive events.

$\therefore$ $0 \le \frac{1}{5}(1+5p) + \frac{1}{3}(1+2p) + \frac{1}{3}(1-p) + \frac{1}{5}(1-3p) \le 1$

i.e. $0 \le \frac{16+11p}{15} \le 1 \Rightarrow 0 \le 16+11p \le 15$

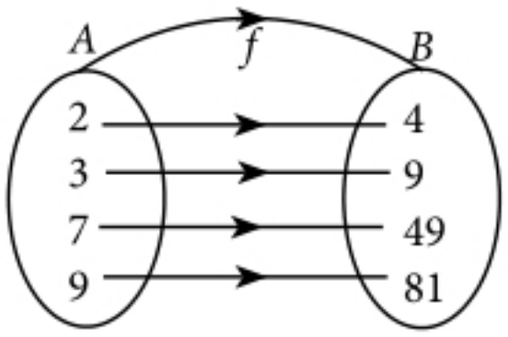
$\Rightarrow$ $-16 \le 11p \le -1 \Rightarrow -\frac{16}{11} \le p \le -\frac{1}{11}$

Thus p can take infinitely many values.

17. (b) : The mapping $f: A \to B$ is given by $f(x) = x^2$

$\therefore$ $f(2) = 4, f(3) = 9, f(7) = 49, f(9) = 81$

Clearly, each element in A has a unique image in B, hence, f is a mapping from A to B.



We see that different elements of A have different image in B under mapping f. Therefore f is a one-one mapping.

Also we see that in range of the mapping $f: A \to B$, given by $f(x) = x^2$ is equal to its co-domain B *i.e.* $f(A) = B$. Hence f is an onto mapping.

Thus the above mapping is one-one onto. Hence it is bijective.

18. (a) : $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} + f(x) \right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} + \underset{x \to 0}{\text{Lt}} f(x) = 1 + \underset{x \to 0}{\text{Lt}} f(x)$$

Thus the given limit will not exist if $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

19. (b) : $f: R \to R$ is a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$

Clearly $f(x) = \frac{1}{3}$ when $x = 0$

Thus the statement-1 is true.

Again $f(x) > 0 \ \forall \ x \in R$

and $f(x)$ will be maximum when $e^x + 2e^{-x}$ is minimum.

Since $e^x + 2e^{-x} \ge 2\sqrt{e^x \cdot 2e^{-x}} \ge 2\sqrt{2}$

$\therefore$ The minimum value of $e^x + 2e^{-x}$ is $2\sqrt{2}$

and maximum value of $f(x)$ is $\frac{1}{2\sqrt{2}}$

$\therefore$ $0 < f(x) \le \frac{1}{2\sqrt{2}}$

Thus the statement-2 is true.

20. (c): 4 numbers can be drawn from 20 numbers in ${}^{20}C_4$ ways

$\therefore$ Exhaustive cases = ${}^{20}C_4$

Favourable cases may be enumerated as below.

When common difference is 1: (1, 2, 3, 4), (2, 3, 4, 5), ..., ..., (17, 18, 19, 20)

When common difference is 2: (1, 3, 5, 7), (2, 4, 6, 8), ..., ..., (14, 16, 18, 20)

When common difference is 3: (1, 4, 7, 10), (2, 5, 8, 11), ..., (11, 14, 17, 20)

When common difference is 4: (1, 5, 9, 13), (2, 6, 10, 14), ..., (8, 12, 16, 20)

When common difference is 5: (1, 6, 11, 16), (2, 7, 12, 17), ..., (5, 10, 15, 20)

When common difference is 6: (1, 7, 13, 19), (2, 8, 14, 20), ...,

$\therefore$ Total number of A.P's with common difference 1, 2, 3, 4, 5 and 6 are $(17 + 14 + 11 + 8 + 5 + 2) = 57$

$\therefore$ Required probability $= \dfrac{57}{{}^{20}C_4} = \dfrac{1}{85}$

So the statement-2 is false and statement-1 is true.

21. (1) : There are four letters and four envelopes. The letters are placed in correctly addressed envelopes with a probability $\dfrac{1}{4!} = \dfrac{1}{24}$

[$\because$ total number of ways = 4! and favourable number of ways = 1]

22. (3) : Given that A and B are two independent events such that $P(A) = \dfrac{3}{8}, P(B) = \dfrac{1}{2}$

$$P(A \cap B) = \frac{1}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{3+4-2}{8} = \frac{5}{8}$$

$\therefore \; P(A \cup B)^c = 1 - P(A \cup B) = 1 - \dfrac{5}{8} = \dfrac{3}{8}$

$\therefore \; K = 3$

23. (8) : The function $f(x) = \dfrac{64(\sqrt{x+4} - 2)}{\sin 2x}$ will be continuous at $x = 0$

if $\lim\limits_{x \to 0} \dfrac{64(\sqrt{x+4} - 2)}{\sin 2x} = f(0)$

$$\Rightarrow \; 64 \lim_{x \to 0} \frac{\sqrt{x+4} - 2}{2x} \times \lim_{x \to 0} \frac{2x}{\sin 2x} = f(0)$$

$$\Rightarrow \; 64 \lim_{x \to 0} \frac{x}{2x(\sqrt{x+4} + 2)} \cdot 1 = f(0)$$

$$\Rightarrow \; 32 \lim_{x \to 0} \frac{1}{\sqrt{x+4} + 2} = f(0)$$

$$\Rightarrow \; 32 \times \frac{1}{4} = f(0) \Rightarrow \; f(0) = 8$$

24. (7) : Here $f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$

Putting $x = 1$ and $y = 2$ on both sides, we get

$f(1) + f(2 \times 1 + 2) + 5(1)(2) = f(3 \times 1 - 2) + 2 \cdot 1^2 + 1$

$\Rightarrow \; f(1) + f(4) + 10 = f(1) + 3$

$\Rightarrow \; f(4) = -7 \Rightarrow |f(4)| = 7$

25. (2) : $\lim\limits_{x \to \pi/4} \dfrac{8 - \sqrt{2}\,(\cos x + \sin x)^5}{1 - \sin 2x}$

$$= \lim_{x \to \pi/4} \frac{8 - (\sqrt{2})^6 \cos^5\left(\frac{\pi}{4} - x\right)}{1 - \sin 2x}$$

$$= 8 \lim_{x \to \pi/4} \frac{1 - \cos^5\left(\frac{\pi}{4} - x\right)}{1 - \sin 2x}$$

$$= 8 \lim_{z \to 0} \frac{1 - \cos^5 z}{1 - \cos 2z} \quad \text{put } \frac{\pi}{4} - x = z$$

$$= 8 \lim_{z \to 0} \frac{5\cos^4 z \sin z}{2 \sin 2z} \text{ [Applying L' Hospital rule]}$$

$$= \frac{40}{4} \lim_{z \to 0} \frac{\cos^4 z \sin z}{\sin z \cos z} = 10 \lim_{z \to 0} \cos^3 z = 10 \cdot 1 = 10$$

PAPER-II

1. (a) : $4\{x\} = x + [x] = [x] + \{x\} + [x]$

$\Rightarrow \; 3\{x\} = 2\,[x] \Rightarrow \; \{x\} = \dfrac{2}{3}[x]$

Now since $\{x\} < 1 \;\; \therefore \;\; [x] = 1 \;\; \therefore \;\; \{x\} = \dfrac{2}{3}$

Now $x = [x] + \{x\} = 1 + \dfrac{2}{3} = \dfrac{5}{3}$

2. (c): $y = x^2 - px + x + 9$

$$= x^2 - (p - 1)\,x + 9$$

$$= x^2 - 2 \cdot x \cdot \frac{p-1}{2} + \left(\frac{p-1}{2}\right)^2 + 9 - \left(\frac{p-1}{2}\right)^2$$

$$= \left(x - \frac{p-1}{2}\right)^2 + 9 - \left(\frac{p-1}{2}\right)^2$$

If the graph of the function is above the x-axis, then $y > 0$

$\because \; \left(x - \dfrac{p-1}{2}\right)^2 \geq 0 \;\; \therefore \;\; 9 - \left(\dfrac{p-1}{2}\right)^2 > 0$

$$\Rightarrow \; 9 > \left(\frac{p-1}{2}\right)^2 \Rightarrow \; (p-1)^2 < 36$$

i.e. $|p - 1| < 6 \Rightarrow -6 < p - 1 < 6 \Rightarrow -5 < p < 7$

3. (c): $f(x) = \sqrt{x(2-x)} \cdot \log(x-1)$

Clearly $f(x)$ will be defined, when

$x(2 - x) \geq 0$ and $x - 1 > 0$

Now, $x(2 - x) \geq 0 \Rightarrow \; x(x - 2) \leq 0 \Rightarrow 0 \leq x \leq 2$

Again $x - 1 > 0 \Rightarrow x > 1$

Thus the above two conditions will be satisfied in $1 < x \leq 2$.

4. (a) : Given that $f(x)$ is an even function and $g(x)$ is an odd function.

$\therefore \quad f(-x) = f(x)$ and $g(-x) = -g(x)$

$$\text{Now } x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x) \qquad \ldots(i)$$

$$\therefore \quad x^2 f(-x) - 2f\left(-\frac{1}{x}\right) = g(-x)$$

$$\Rightarrow \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = -g(x) \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2x^2 f(x) - 4f\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = 0 \qquad \ldots(iii)$$

Replacing x by $\frac{1}{x}$, we get

$$\frac{1}{x^2} f\left(\frac{1}{x}\right) - 2f(x) = 0 \qquad \ldots(iv)$$

From (iii) and (iv), $f(x) = 0 \quad \therefore \quad f(10) = 0 \quad (\because \ x \neq 0)$

5. (b) : $f(x) = \sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x$

$$= 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x$$

$\because$ $\cos x$ is a periodic function, whose least period is 2π.

$$\therefore \quad \cos 4x = \cos(2\pi + 4x) = \cos\left\{4\left(x + \frac{\pi}{2}\right)\right\}$$

$\therefore$ The least period of the function is $\frac{\pi}{2}$.

6. (d) :
$$\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x-1)}}{x - 1}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\,|\sin h|}{h} \quad \{\text{put } (x-1) = h\}$$

$$= \sqrt{2} \lim_{h \to 0} \frac{\sin h}{h}$$

$$\text{Now, } \lim_{h \to 0^+} \frac{|\sin h|}{h} = 1 \Rightarrow \lim_{h \to 0^-} \frac{|\sin h|}{h} = -1$$

$\because$ L.H.L. $\neq$ R.H.L

$\therefore$ The limit of the function does not exist.

7. (c)

8. (c): The number of ways in which the American man and wife are sit together is $(6 - 1)!\ 2! \times 2! \times 2! \times 2!$

i.e. $5!\ (2!)^4$

The number of ways in which all men and wives are sit together $= (5 - 1)! \times 2! \times 2! \times 2! \times 2! \times 2!$

$$= 4! \times (2!)^5$$

$\therefore$ Required conditional probability

$$= \frac{4! \times (2!)^5}{5! \times (2!)^4} = \frac{2}{5}$$

9. (c)

10. (c): Let A be the event that the man selected the winning horse.

$\therefore$ Exhaustive cases $= {}^5C_2$

Favourable cases $= 1 \times {}^4C_1$

$$\therefore \quad P(A) = \frac{1 \times {}^4C_1}{{}^5C_2} = \frac{4}{10} = \frac{2}{5}$$

11. (a, b, c)

12. (a, b, c, d) : $f(x) = 2x + 3$, when $x \leq 1$

$= ax^2 + bx$, when $x > 1$

$\because$ $f(x)$ is differentiable everywhere

$\therefore$ $f(x)$ is continuous everywhere

$\therefore$ $f(x)$ is continuous at $x = 1$

$$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{x \to 1} (2x + 3) = \lim_{x \to 1} (ax^2 + bx) = (2x + 3)_{x=1}$$

$$\Rightarrow \quad 5 = a + b = 5 \quad \therefore \quad a + b = 5 \qquad \ldots(i)$$

Again, $f(x)$ is differentiable at $x = 1 \quad \therefore \quad f'(1)$ exists

i.e. $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ exists

$$\therefore \quad \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{2(1+h) + 3 - 5}{h}$$

$$= \lim_{h \to 0} \frac{a(1+h)^2 + b(1+h) - 5}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} \frac{ah^2 + 2ah + bh + a + b - 5}{h}$$

$$\Rightarrow \quad 2 = \lim_{h \to 0} (ah + 2a + b) \ [\because \ h \neq 0 \text{ and } a + b = 5]$$

$$\Rightarrow \quad 2 = 2a + b \qquad \ldots(ii)$$

Now solving (i) and (ii), we get $a = -3$, $b = 8$

$\therefore$ $f(x) = -3x^2 + 8x$ when $x > 1$

$\therefore$ $f'(x) = -6x + 8$, when $x > 1$

$\therefore$ $f'(2) = -12 + 8 = -4$

13. (b, c) : $f(x)=\dfrac{\sin([x]\pi)}{x^2+x+1}=0\ \forall\ x\in R$

$\therefore$ $f(x)$ is a constant function

$\therefore$ f is many one and into function.

14. (b, c): $A+B+C+D+E+F+G=0.75$

$D+E+F+G=0.5$

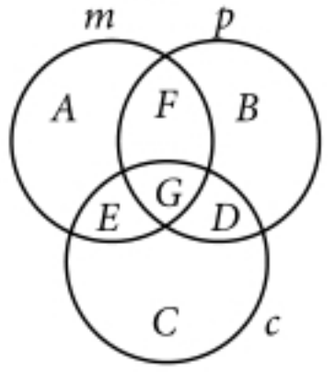
& $D+E+F=0.4$

$\Rightarrow$ $G=0.1,\ A+B+C=0.25$

$m=A+E+F+G;$

$p=B+D+F+G;$

$c=C+D+E+G$



$\therefore$ $m+p+c=A+B+C+2(D+E+F)+3G$

$=0.25+2(0.4)+3(0.1)=1.35=\dfrac{27}{20}$

$mpc=G=0.1=\dfrac{1}{10}.$

15. (a, c) : Probability that exactly one of the independent events A and B occurs

$=P[(A\cap B^c)\cup(A^c\cap B)]$

$=P(A\cap B^c)+P(A^c\cap B)$

[$\because$ $A\cap B^c$ and $A^c\cap B$ are mutually exclusive events]

$=P(A)\,P(B^c)+P(A^c)\,P(B)$

$=P(A)-P(A)P(B)+P(B)-P(A)P(B)$

$=P(A)+P(B)-2P(A)P(B)$

$=1-P(A^c)+1-P(B^c)-2\{1-P(A^c)\}\{1-P(B^c)\}$

$=1-P(A^c)+1-P(B^c)-2\,(1-P(A^c)-P(B^c)+P(A^c)P(B^c))$

$=P(A^c)+P(B^c)-2P(A^c)P(B^c)$

$=P(A^c)+P(B^c)-2P(A^c\cap B^c)$

16. (a) : Let E be the event of getting a six when a die is thrown

$\therefore$ $P(E)=\dfrac{1}{6}\ \Rightarrow\ P(E^c)=1-\dfrac{1}{6}=\dfrac{5}{6}$

$\therefore$ Required probability $=P(E^c)\cdot P(E^c)P(E)$

$=\dfrac{5}{6}\cdot\dfrac{5}{6}\cdot\dfrac{1}{6}=\dfrac{25}{216}$

17. (d) : Required probability

$=P(E^c)P(E^c)P(E)+P(E^c)P(E^c)P(E^c)P(E)+\ldots\ldots$

$=\left(\dfrac{5}{6}\right)^2\cdot\dfrac{1}{6}+\left(\dfrac{5}{6}\right)^3\cdot\dfrac{1}{6}+\left(\dfrac{5}{6}\right)^4\cdot\dfrac{1}{6}+\ldots\ldots\ldots$

$=\left(\dfrac{5}{6}\right)^2\cdot\dfrac{1}{6}\left\{1+\left(\dfrac{5}{6}\right)+\left(\dfrac{5}{6}\right)^2+\ldots\ldots\text{ to }\infty\right\}$

$=\left(\dfrac{5}{6}\right)^2\cdot\dfrac{1}{6}\cdot\dfrac{1}{1-\dfrac{5}{6}}=\dfrac{25}{36}$

18. (d) : $P(X\geq 6\mid X>3)$

$=P(X\geq 6\mid X\geq 4)$

$=\dfrac{\left(\dfrac{5}{6}\right)^5\cdot\dfrac{1}{6}\left\{1+\dfrac{5}{6}+\left(\dfrac{5}{6}\right)^2+\ldots\ldots\text{ to }\infty\right\}}{\left(\dfrac{5}{6}\right)^3\cdot\dfrac{1}{6}\left\{1+\dfrac{5}{6}+\left(\dfrac{5}{6}\right)^2+\ldots\ldots\text{ to }\infty\right\}}$

$=\left(\dfrac{5}{6}\right)^2=\dfrac{25}{36}$

19. (b) : The given relation is,

$(x^2-y^2)\,f(x)f(y)=xy\,\{(x-y)f(x+y)+(x+y)f(x-y)\}$ ...(i)

Put $y=0$ in (i), we get

$(x^2-0^2)\,f(x)\,f(0)=0$

$\Rightarrow$ $x^2f(x)\,f(0)=0$

But $x\neq 0, f(x)\neq 0$ because $f'(x)$ exists.

$\therefore$ $f(0)=0$

So, $f(x)=2x$, which is an odd function.

20. (d) : $f(\ln 2)=2\ln 2$

21. (b) : $\underset{x\to 0}{\text{Lt}}\left[\dfrac{f(x)}{\sin x}\right]=\underset{x\to 0}{\text{Lt}}\left[\dfrac{2x}{\sin x}\right]=2$

22. (A) → p, q; (B) → s; (C) → p, q; (D) → p, q, s

(A) : $f(x)=\underset{x\to\infty}{\text{Lt}}\dfrac{x^{2n}-1}{x^{2n}+1}=\underset{x\to\infty}{\text{Lt}}\dfrac{1-\dfrac{1}{x^{2n}}}{1+\dfrac{1}{x^{2n}}}=\dfrac{1-0}{1+0}$

$\therefore$ $f(x)=1$ on $(1,\infty),(2,\infty)$

(B) : Clearly $f(x)=\underset{x\to\infty}{\text{Lt}}\dfrac{x^{2n}-1}{x^{2n}+1}=-1$ when $x\in\left(\dfrac{-1}{2},\dfrac{1}{2}\right)$

(D) : Again $f(x)=\text{sgn}\,(|x|-1)$

$\text{sgn}\,(|x|-1)=-1$ when $|x|-1<0$

$\Rightarrow$ $|x|<1\ \Rightarrow\ x\in(-1,1)$

$\text{sgn}\,(|x|-1)=0$ when $|x|-1=0\ \Rightarrow\ |x|=1$

$\Rightarrow$ $x=\pm 1$

$\text{sgn}\,(|x|-1)=1$ when $|x|-1>0\ \Rightarrow\ |x|>1$

i.e. $x>1$ or $x<-1$

23. (A) → p, q, r; (B) → p, r, s; (C) → p, r, s; (D) → p, r, s

ENRICH YOUR CONCEPTS ON XI

Straight Lines, Conic Sections & Introduction to Three Dimensional Geometry

Based on NCERT Pattern

Series-6

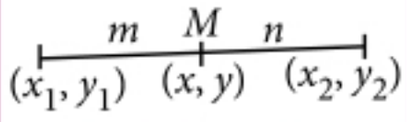
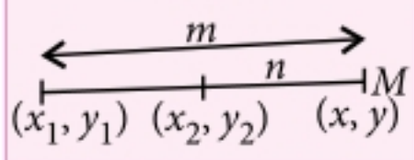
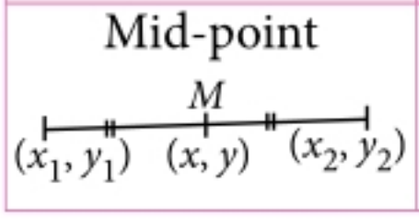
STRAIGHT LINES

Definition : A path traced by a point in a constant direction and endlessly in its opposite direction is known as a straight line.

- **Distance between two points :** The distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Section Formula :** Let a point M divides a line segment joining the points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$, then

Types of division	Coordinates of M
Internal 	$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$
External 	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$
Mid-point 	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- **Area of Triangle :** Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

Note : (i) If area = 0, then the three points are said to be collinear.

(ii) Area is a non-negative quantity.

- **Slope of a Line :** If a line makes an angle θ $(0 \le \theta < 180°)$ with positive direction of x-axis, then θ is called the inclination of the line and $\tan\theta$ is called the slope of line and it is denoted by m.

Slope of a line in different conditions :

(i) Slope of a horizontal line is 0.

(ii) Slope of a vertical line is not defined.

(iii) If the equation of line is given, then its slope (m) is given by

$$m = \frac{-\text{ coefficient of } x}{\text{coefficient of } y}$$

(iv) Slope of a line passing through (x_1, y_1) and (x_2, y_2), is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Conditions for Parallelism and Perpendicularity:

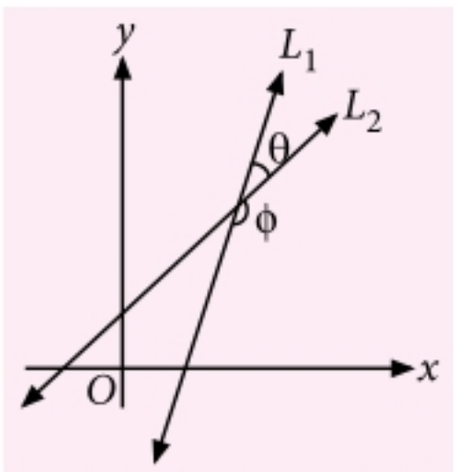
Let l_1 and l_2 be two straight lines and m_1, m_2 respectively be the slopes, then

(i) l_1 is parallel to l_2 if $m_1 = m_2$

(ii) l_1 is perpendicular to l_2 if $m_1 m_2 = -1$

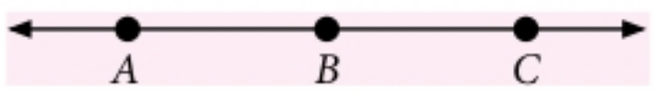
Angle between Two Lines : Angle θ between the two lines l_1, l_2 with slopes m_1, m_2 respectively is

$$\theta = \tan^{-1}\left(\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\right)$$



Remarks : If $\tan\theta > 0 \Rightarrow \tan\phi < 0$, then θ will be acute and ϕ will be obtuse and vice versa.

Collinearity of three points : Three points A, B, C in a plane are said to be collinear, if and only if slope of AB = slope of BC.



Equation of Line in Different Forms :

Form	Equation of Line
x-axis	$y = 0$
y-axis	$x = 0$
Horizontal line	$y = a$
Vertical line	$x = b$
Point slope form	$y - y_1 = m\,(x - x_1)$ Slope = m, passing through point (x_1, y_1)
Two point form	$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ Passing through points (x_1, y_1) and (x_2, y_2)
Slope Intercept form	$y = mx + c$ Slope = m, c = y-intercept
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$ x-intercept = a, y-intercept = b
Normal form	$x \cos\alpha + y \sin\alpha = p$ p = The distance of the normal from origin to the line, α = The angle between the normal and the positive direction of x-axis.
Distance form	$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ r = The distance between (x, y) and (x_1, y_1), θ = Angle with the positive direction of x-axis.

General Equation of a Line :

$$Ax + By + C = 0$$

where A, B and C are real constants such that A and B are not zero simultaneously.

Different Forms of General Equation of a Line :

Let the given equation of the line be

$$Ax + By + C = 0$$

Form	Equation of Line
Slope Intercept form	$y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$ $m = \frac{-A}{B}$ and $c = \frac{-C}{B}$.
Intercept form	$\frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1$ $a = \frac{-C}{A}$ and $b = \frac{-C}{B}$.
Normal form	$\left(\frac{-A}{\sqrt{A^2+B^2}}\right)x + \left(\frac{-B}{\sqrt{A^2+B^2}}\right)y = \frac{C}{\sqrt{A^2+B^2}}$ $\cos\alpha = \frac{-A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{-B}{\sqrt{A^2+B^2}}$ and $p = \frac{C}{\sqrt{A^2+B^2}}$.

Distance of a point from a line :

The length of the perpendicular from a point (x_1, y_1) to a line $Ax + By + C = 0$ is

$$\left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$$

Distance between two parallel lines :

Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is

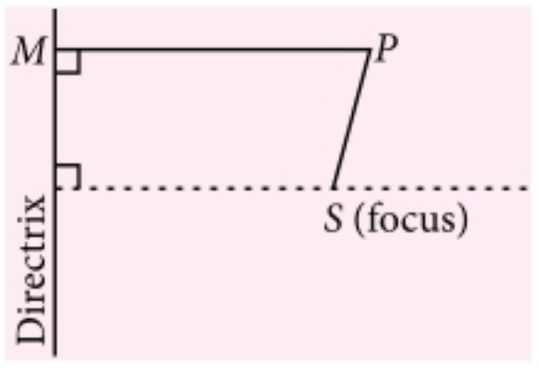
$$\frac{|C_1 - C_2|}{\sqrt{A^2+B^2}}$$

CONIC SECTIONS

Definition : The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a **conic section** or a **conic**.

The fixed point is called **focus** of the conic, fixed line is called **directrix** of the conic and the constant ratio is called **eccentricity** of the conic.

If $e = 1$, the conic is called **Parabola**.
If $e < 1$, the conic is called **Ellipse**.
If $e > 1$, the conic is called **Hyperbola**.



If $e = 0$, the conic is called **Circle**.
If $e = \infty$, the conic is called **Pair of straight lines**.

CIRCLES

Definition : A circle is the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant.

The fixed point is called the centre of the circle and the constant distance is known as the radius of the circle.

Equation of circle :

(i) **Standard form :** $x^2 + y^2 = a^2$.

Centre = (0, 0) and Radius = 'a' units.

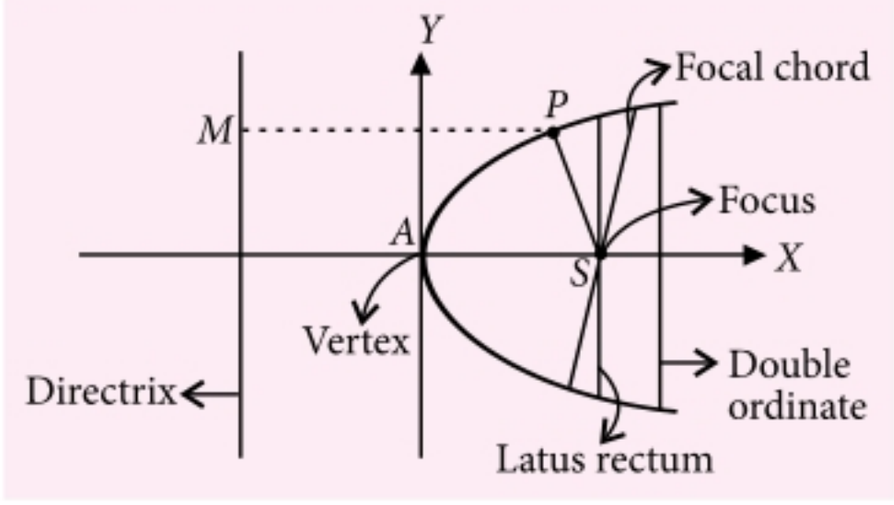
(ii) **General form :** $x^2 + y^2 + 2gx + 2fy + c = 0$

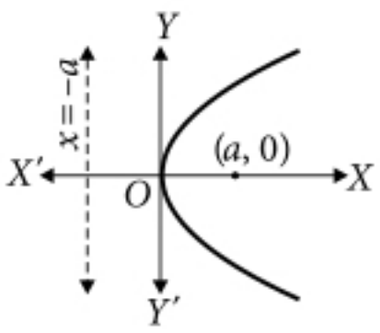
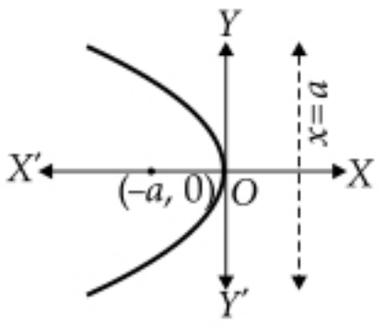
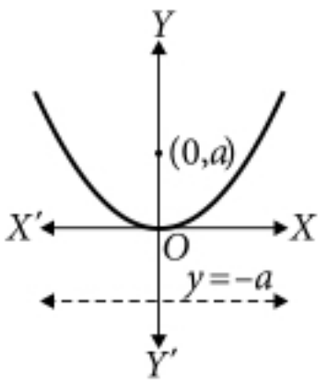
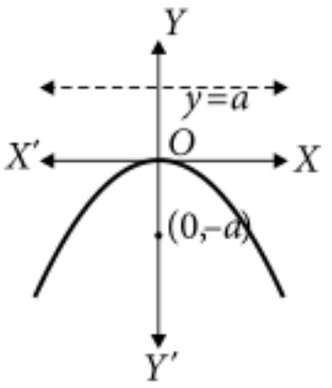
Centre = $(-g, -f)$ and Radius = $\sqrt{g^2 + f^2 - c}$ units.

(iii) **With centre (h, k) and radius r :**

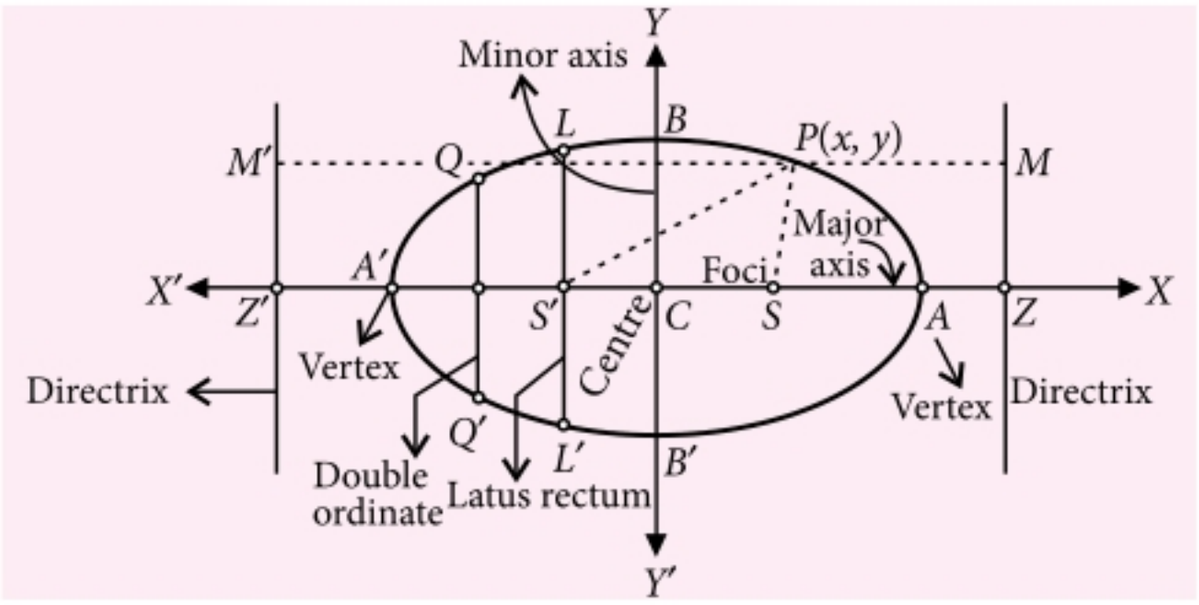
$$(x - h)^2 + (y - k)^2 = r^2$$

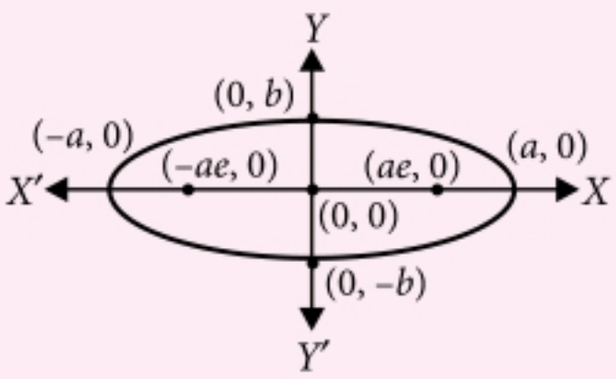
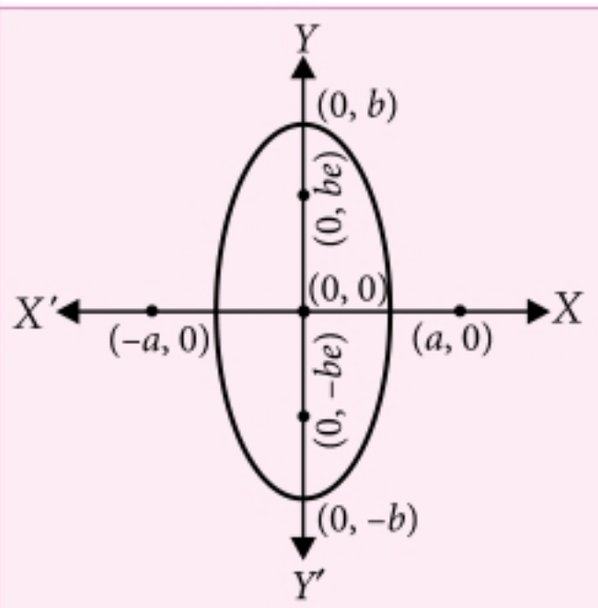
PARABOLA



Equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graphs				
Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Extremities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

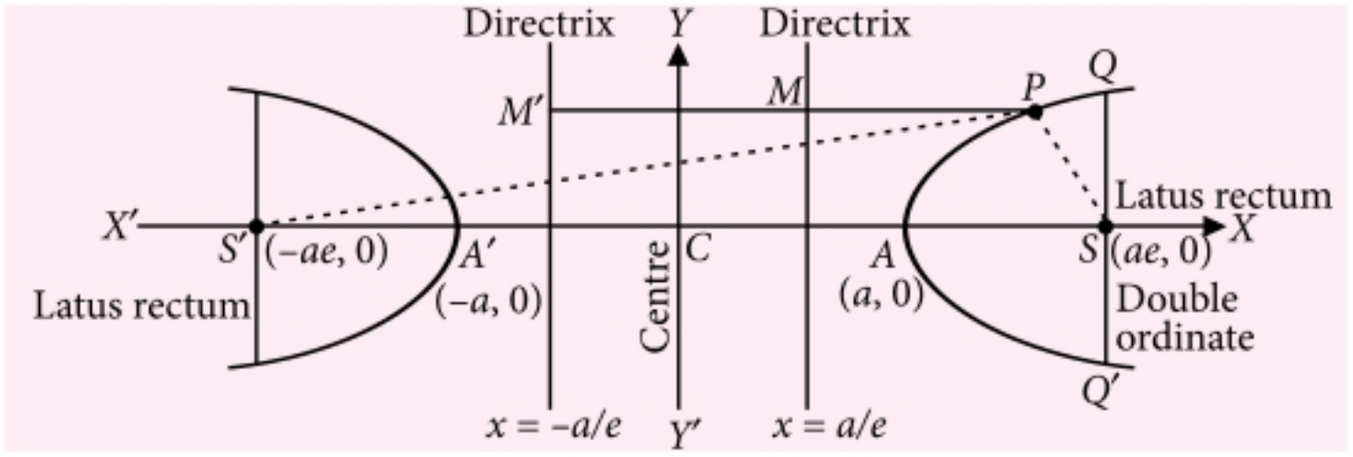
ELLIPSE

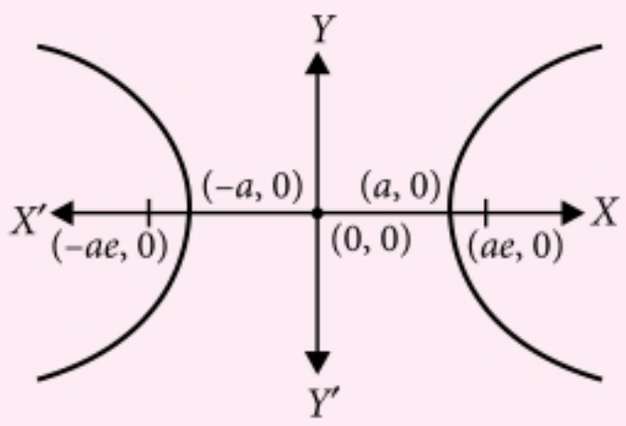
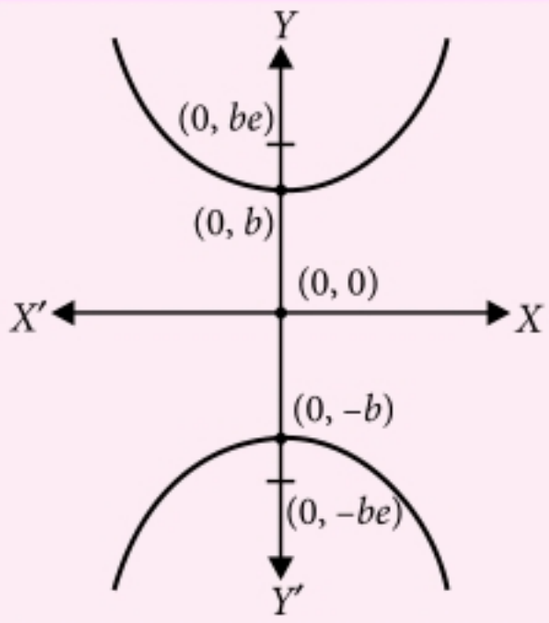


Fundamental terms	**Ellipse** (Horizontal ellipse)	**Conjugate Ellipse** (Vertical ellipse)
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$
Graph		
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distance or radii	$\lvert SP\rvert = (a - ex_1)$ and $\lvert S'P\rvert = (a + ex_1)$	$\lvert SP\rvert = (b - ey_1)$ and $\lvert S'P\rvert = (b + ey_1)$

Sum of focal radii $\lvert SP\rvert + \lvert S'P\rvert$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

HYPERBOLA



Fundamental terms	**Hyperbola**	**Conjugate Hyperbola**
Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Graph		
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$

Difference of focal radii $= \|SP\| - \|S'P\|$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

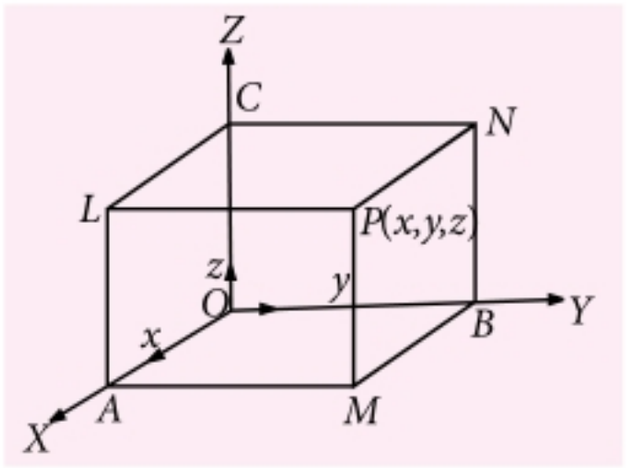
INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

COORDINATE SYSTEM

The coordinate system in space is called a three dimensional system. The planes YOZ, ZOX and XOY are respectively known as yz-plane, zx-plane and xy-plane.

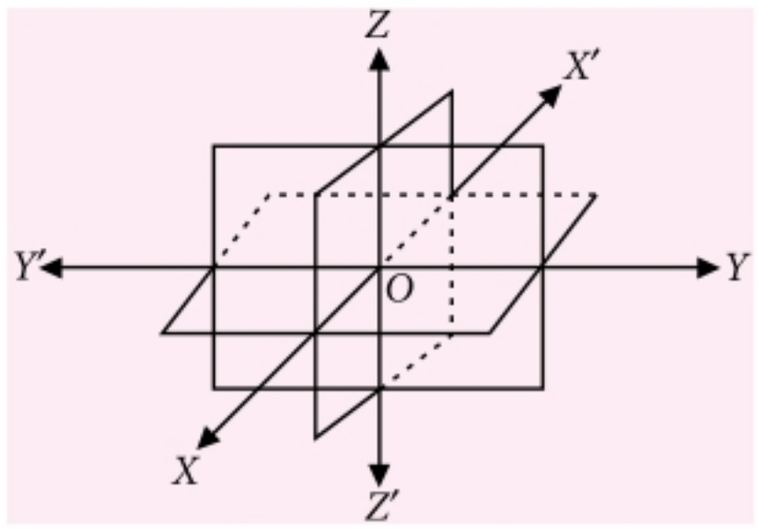
These planes, known as coordinate planes, divide the space into eight parts, called *octants*.

Coordinates of a Point : Let P be a point in space and $OA = x$, $OB = y$ and $OC = z$. Then, the coordinates of P are (x, y, z).



The coordinates of origin O are $(0, 0, 0)$.
The coordinates of any point on the x-axis is $(x, 0, 0)$, on y-axis is $(0, y, 0)$ and on z-axis is $(0, 0, z)$. The coordinates of any points in yz-plane be $(0, y, z)$, in xy-plane be $(x, y, 0)$ and in zx-plane be $(x, 0, z)$.

Signs of Coordinates in Eight Octants :



These octants could be named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ and denoted by I, II, III, ..., VIII respectively.

	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

Distance between two points : Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points, then distance between P and Q is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formula : Let M be a point which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$, then

Types of division	Coordinates of M
Internal	$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$
External	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

Mid-point Formula : Coordinates of mid-point of a line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is,

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

COORDINATES OF CENTROID OF A TRIANGLE

The coordinates of the centroid of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

VERY SHORT ANSWER TYPE (1 MARK)

1. The vertices of ΔPQR are $P(2, 1)$, $Q(-2, 3)$ and $R(4, 5)$. Find the points of the median through the vertex R.
2. Find the equation of the circle with centre $(5, -2)$ and radius 3.
3. Find the equation of a straight line parallel to x-axis and at a distance 3 units below it.
4. Find the focus, equation of directrix and eccentricity of the conic represented by the equation $y^2 = 2\sqrt{3}\,x$.

5. If the origin is the centroid of the triangle PQR with vertices $P(2a, 2, 6)$, $Q(-4, 3b, -10)$ and $R(8, 14, 2c)$, then find the values of a, b and c.

SHORT ANSWER TYPE (4 MARKS)

6. Find the ratio in which the line segment joining the points (2, 4, 5) and (3, 5, –4) is divided by the xz-plane.

7. Find the equation of a circle passing through the points (5, 7), (6, 6) and (2, –2). Find its centre and radius.

8. Find the points of trisection of the segment joining the points $P(1, 0, -6)$ and $Q(-5, 9, 6)$.

9. A circle has radius 3 units and its centre lies on the line $y = x - 1$. If circle passes through the point (7, 3), find the equation of the circle.

10. The midpoints of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find its vertices.

LONG ANSWER TYPE (6 MARKS)

11. Find the equations of the two lines which can be drawn through the point (2, 2) to make an angle of 45° with the line $x + y = 2$.

12. Find the equation of the ellipse with centre at the origin, major axis on the y-axis and passing through the points (3, 2) and (1, 6).

13. $\angle AOB$ is a right angle, where O is the origin and $OA = OB = \sqrt{8}$ units. If OA makes an angle of 45° with the x-axis, find the coordinates of B.

14. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

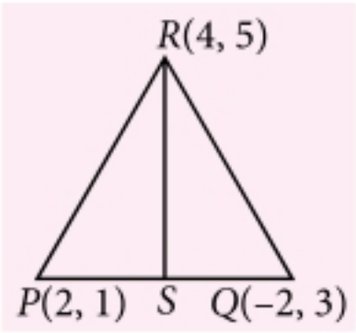
15. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, find the equation of the hyperbola if its eccentricity is 2.

SOLUTIONS

1. Let RS be the median, then S is the mid point of PQ.

$\therefore$ Coordinates of S

$$=\left(\frac{2-2}{2}, \frac{1+3}{2}\right)=(0, 2)$$



2. Here $h = 5$, $k = -2$ and $r = 3$

$\therefore$ The equation of the circle is,

$$(x-5)^2+(y+2)^2=(3)^2$$

$$\Rightarrow x^2+25-10x+y^2+4+4y=9$$

$$\Rightarrow x^2+y^2-10x+4y+20=0$$

3. We know that the equation of any line parallel to x-axis is $y = b$. Here, $b = -3$.

$\therefore$ Required equation is $y = -3$ or $y + 3 = 0$.

4. The given conic $y^2 = 2\sqrt{3}x$ is a parabola.

It is comparable with $y^2 = 4ax$

Hence, eccentricity is 1, focus $= \left(\frac{\sqrt{3}}{2}, 0\right)$ and equation of directrix is $x+\frac{\sqrt{3}}{2}=0$.

5. Coordinates of centroid of ΔPQR is

$$\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)$$

$$=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

But it is given that coordinates of centroid is (0, 0, 0)

$$\therefore \quad \frac{2a+4}{3}=0 \Rightarrow a=-2$$

$$\frac{3b+16}{3}=0 \Rightarrow b=\frac{-16}{3}$$

$$\frac{2c-4}{3}=0 \Rightarrow c=2$$

6. Let the joint of $P(2, 4, 5)$ and $Q(3, 5, -4)$ be divided by xz-plane in the ratio $k : 1$ at the point $R(x, y, z)$.

$$\therefore \quad x=\frac{3k+2}{k+1}, \; y=\frac{5k+4}{k+1}, \; z=\frac{-4k+5}{k+1}$$

Since the point $R(x, y, z)$ lies on the xz-plane, the y-coordinate should be zero, *i.e.*,

$$\frac{5k+4}{k+1}=0 \Rightarrow k=-\frac{4}{5}$$

7. Let the general equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

Since it passes through the points (5, 7), (6, 6) and (2, –2), each one of these points must satisfy (i).

$\therefore$ $25 + 49 + 10g + 14f + c = 0$

$\Rightarrow$ $10g + 14f + c + 74 = 0$ ...(ii)

$36 + 36 + 12g + 12f + c = 0$

$\Rightarrow 12g + 12f + c + 72 = 0$ ...(iii)

$4 + 4 + 4g - 4f + c = 0$

$\Rightarrow$ $4g - 4f + c + 8 = 0$ ...(iv)

Solving (ii) and (iii) we get,

$g - f = 1$ ...(v)

Solving (iii) and (iv) we get,

$g + 2f = -8$ ...(vi)

Solving (v) and (vi), we get $g = -2$ and $f = -3$

$\therefore$ From (ii), we get $c = -12$

$\therefore$ From (i), we get

$$x^2 + y^2 - 4x - 6y - 12 = 0,$$

which is the required equation of circle.

Centre of the circle $= (-g, -f) = (2, 3)$

And, radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4+9+12}$

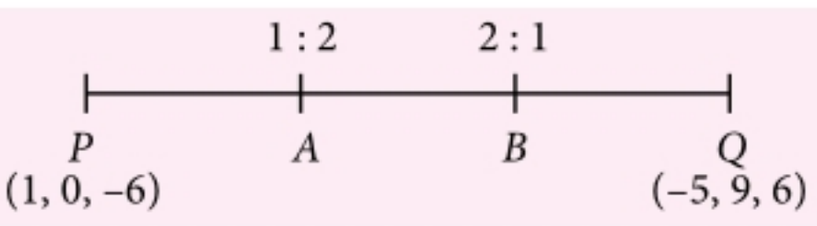
$= \sqrt{25} = 5$ units.

8. Let A and B be the points of trisection of the segment PQ, then A divides PQ in the ratio $1 : 2$ and B divides PQ in the ratio $2 : 1$.

$\therefore$ Coordinates of A

$$\equiv\left(\frac{1\times(-5)+2\times1}{1+2}, \frac{1\times9+2\times0}{1+2}, \frac{1\times6+2\times(-6)}{1+2}\right)$$

i.e., A is $(-1, 3, -2)$



Also, coordinates of B

$$\equiv\left(\frac{2\times(-5)+1\times1}{2+1}, \frac{2\times9+1\times0}{2+1}, \frac{2\times6+1\times(-6)}{2+1}\right),$$

i.e., B is $(-3, 6, 2)$

Hence, the required points of trisection are $A(-1, 3, -2)$ and $B(-3, 6, 2)$.

9. Let (h, k) be the centre of the circle, then $k = h - 1$.

Therefore, the equation of the circle is

$(x - h)^2 + [y - (h - 1)]^2 = 9$ ... (i)

Given that the circle passes through the point $(7, 3)$ and hence, we get

$(7 - h)^2 + [3 - (h - 1)]^2 = 9$

or $(7 - h)^2 + (4 - h)^2 = 9$

or $h^2 - 11h + 28 = 0$

which gives $(h - 7)(h - 4) = 0$

$\Rightarrow$ $h = 4$ or $h = 7$.

Therefore, the required equations of the circles are $x^2 + y^2 - 8x - 6y + 16 = 0$

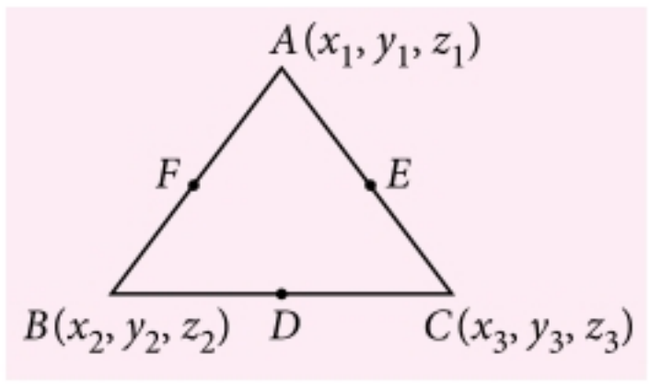
or $x^2 + y^2 - 14x - 12y + 76 = 0$ {using (i)}

10. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the mid points of the sides BC, CA and AB respectively. Then,

$$\frac{x_2 + x_3}{2} = 1;$$

$$\frac{y_2 + y_3}{2} = 5;$$

$$\frac{z_2 + z_3}{2} = -1;$$



$\Rightarrow$ $x_2 + x_3 = 2$ ...(i), $y_2 + y_3 = 10$ ...(ii),

$z_2 + z_3 = -2$ ...(iii)

Similarly,

$x_3 + x_1 = 0$ ...(iv), $y_3 + y_1 = 8$ ...(v),

$z_3 + z_1 = -4$ ...(vi)

And,

$x_1 + x_2 = 4$ ...(vii), $y_1 + y_2 = 6$ ...(viii),

$z_1 + z_2 = 8$ ...(ix)

Adding (i), (iv), (vii), we get,

$2(x_1 + x_2 + x_3) = 6 \Rightarrow x_1 + x_2 + x_3 = 3$.

Thus, $x_1 = 1$, $x_2 = 3$ and $x_3 = -1$.

Adding (ii), (v), (viii), we get,

$2(y_1 + y_2 + y_3) = 24 \Rightarrow y_1 + y_2 + y_3 = 12$

$\therefore$ $y_1 = 2$, $y_2 = 4$ and $y_3 = 6$.

Adding (iii), (vi), (ix), we get,

$2(z_1 + z_2 + z_3) = 2 \Rightarrow z_1 + z_2 + z_3 = 1$

$\therefore$ $z_1 = 3$, $z_2 = 5$ and $z_3 = -7$.

Hence, the vertices of the given triangle are $A(1, 2, 3)$; $B(3, 4, 5)$ and $C(-1, 6, -7)$.

11. The given line is $x + y = 2$ ...(i)

$\therefore$ Slope of (i) $= -\frac{1}{1} = -1$.

Let the slope of a line passing through $(2, 2)$, which makes an angle of $45°$ with line (i), be m, then $\tan 45° = \left|\frac{m-(-1)}{1+m(-1)}\right| \Rightarrow 1 = \left|\frac{m+1}{1-m}\right|$

$\Rightarrow$ $\pm 1 = \frac{m+1}{1-m} \Rightarrow \pm(1-m) = m+1$

$\Rightarrow$ $1 - m = m + 1$, or $-(1 - m) = m + 1$

$\Rightarrow$ $m = 0$ (slope is not defined)

Hence, one of the required lines passing through $(2, 2)$ is $y - 2 = 0\ (x - 2)$ *i.e*; $y = 2$.

The other required line ought to be vertical

as the slope is not defined in that case. Thus, the equation of this line is $x - 2 = 0$.

12. Since the major axis of the ellipse lies on y-axis, so it is a vertical ellipse.
Let the required equation be,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (b>a) \qquad ...(1)$$

Now, it is given that (3, 2) and (1, 6) lies on (1)

$$\therefore \quad \frac{9}{a^2}+\frac{4}{b^2}=1 \ ...(2) \quad \text{and} \quad \frac{1}{a^2}+\frac{36}{b^2}=1 \ ...(3)$$

Putting $\frac{1}{a^2}=m$ and $\frac{1}{b^2}=n$ in (2) and (3) respectively, we get

$$9m + 4n = 1 \qquad ...(4)$$
$$m + 36n = 1 \qquad ...(5)$$

Solving (4) and (5), we get,

$$m=\frac{1}{10} \quad \text{and} \quad n=\frac{1}{40}$$

$$\Rightarrow \frac{1}{a^2}=\frac{1}{10} \text{ and } \frac{1}{b^2}=\frac{1}{40}$$

$$\Rightarrow a^2 = 10 \text{ and } b^2 = 40$$

Putting these values in (1), we get

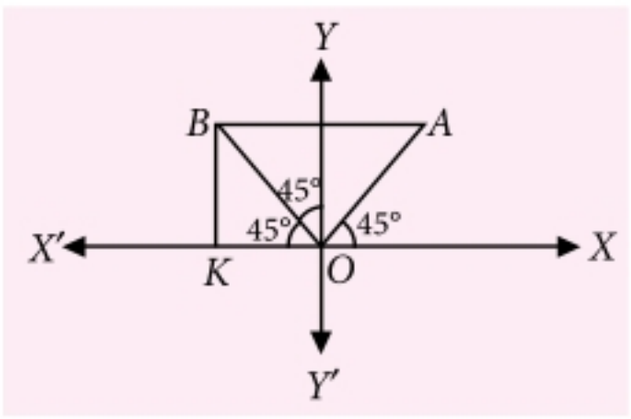
$$\frac{x^2}{10}+\frac{y^2}{40}=1 \text{ or } 4x^2 + y^2 = 40$$

which is the required equation of ellipse.

13. Given that $OA = OB = \sqrt{8}$ units, $\angle AOB = 90°$ and $\angle AOX = 45°$.

$$\Rightarrow \angle AOY = 90° - 45° = 45°$$

$$\Rightarrow \angle BOY = \angle AOB - \angle AOY = 45°$$



Drop perpendicular BK on the negative x-axis from B.

$$\Rightarrow \angle BOK = 45°$$

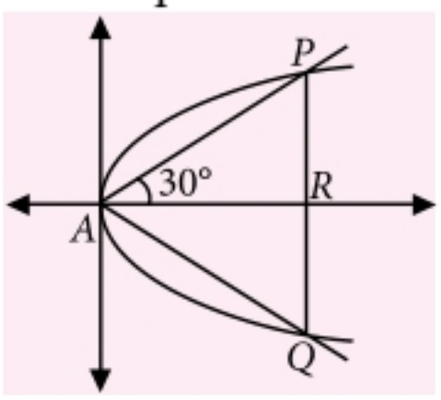
Now, in ΔBOK,

$$BK = OB\sin 45° = \sqrt{8}\times\frac{1}{\sqrt{2}} = 2 \text{ units}$$

$$OK = OB\cos 45° = \sqrt{8}\times\frac{1}{\sqrt{2}} = 2 \text{ units}$$

Coordinates of B are (OK, BK)
But 'B' is in second quadrant. So, coordinates of B is $(-2, 2)$.

14. As shown in the figure APQ denotes the equilateral triangle with its equal sides of length l (say).



Here, $AP = l$

So, $AR = l\cos 30° = \frac{l\sqrt{3}}{2}$

Also, $PR = l\sin 30° = \frac{l}{2}$

Thus, $\left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$ are the coordinates of the point P lying on the parabola $y^2 = 4ax$.

$$\therefore \quad \frac{l^2}{4}=4a\left(\frac{l\sqrt{3}}{2}\right) \Rightarrow l = 8a\sqrt{3}$$

Thus, $8a\sqrt{3}$ is the required length of the side of the equilateral triangle inscribed in the parabola $y^2 = 4ax$.

15. We have, $\frac{x^2}{25}+\frac{y^2}{9}=1$.

$$\Rightarrow a^2 = 25, b^2 = 9 \Rightarrow a = 5, b = 3. \ (\because a, b > 0)$$

If 'e' is the eccentricity of the ellipse, then

$$e=\sqrt{1-\frac{b^2}{a^2}} \Rightarrow e=\sqrt{1-\frac{9}{25}} \Rightarrow e=\frac{4}{5}.$$

Hence, the foci of the ellipse are $(\pm ae, 0)$ and $(-ae, 0)$, *i.e.*, (4, 0) and (– 4, 0).
The required hyperbola has same foci (4, 0) and (–4, 0) which are on the X-axis and are symmetrically placed about the origin
$\Rightarrow$ The axes of the hyperbola coincide with the coordinate axes and its equation can be taken as

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

As the eccentricity of the hyperbola is 2 and foci are (4, 0) and (–4, 0)

$$\therefore \quad ae = 4 \Rightarrow a \times 2 = 4 \Rightarrow a = 2$$

Also $b^2 = a^2 (e^2 - 1) \Rightarrow b^2 = 4 (4 - 1) = 12$

Hence, the required equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$, *i.e.*, $3x^2 - y^2 = 12$.

concept BOOSTERS

Class XI

Permutations and Combinations

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITIONS

- **Permutations :** Each of the arrangements in a definite order which can be made by taking some or all of a number of things is called a permutation.
- **Combinations :** Each of the groups or selections which can be made by taking some or all of a number of things without reference to the order of the things in each group is called a combination.

Fundamental Principle of Counting

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of simultaneous occurrence of both the events in a definite order is $m \times n$. This can be extended to any number of events.

RESULTS :

- A Useful Notation :
  $n! = n\,(n-1)\,(n-2)\ldots\ldots 3\cdot 2\cdot 1$;
  $n! = n\cdot (n-1)!$
  $0! = 1! = 1$; $(2n)! = 2^n \cdot n!\,[1\cdot 3\cdot 5\cdot 7\ldots(2n-1)]$
  Note : factorials of negative integers are not defined.
- If ${}^nP_r$ denotes the number of permutations of n different things, taken r at a time, then
  $${}^nP_r = n\,(n-1)\,(n-2)\ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}.$$
  where $r \le n$ and $n, r \in N$. Note that , ${}^nP_n = n!$.
- If ${}^nC_r$ denotes the number of combinations of n different things taken r at a time, then
  $${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!}$$
  where $r \le n$; $r, n \in N$
- The number of ways in which $(m+n)$ different things can be divided into two groups containing m and n things respectively is $\frac{(m+n)!}{m!n!}$. If $m = n$, the groups are equal and in this case the number of sub-divisions is $\frac{(2n)!}{n!\,n!\,2!}$; for in any one way it is possible to interchange the two groups without obtaining a new distribution. However, if $2n$ things are to be divided equally between two persons, then the number of ways $= \frac{(2n)!}{n!\,n!}$.
- Number of ways in which $(m+n+p)$ different things can be divided into three groups containing m , n and p things respectively is $\frac{(m+n+p)!}{m!\,n!\,p!}$, $m \ne n \ne p$.
- If $m = n = p$, then the number of ways $= \frac{(3n)!}{n!\,n!n!\,3!}$.
  However, if $3n$ things are to be divided equally among three people, then the number of ways $= \frac{(3n)!}{(n!)^3}$.

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He trains IIT and Olympiad aspirants.

- The number of permutations of n things taken all at a time when p of them are similar & of one type, q of them are similar & of another type, r of them are similar and of a third type and the remaining $n-(p+q+r)$ are all different, is $\frac{n!}{p!q!r!}$.
- The number of circular permutations of n different things taken all at a time is $(n-1)!$. If clockwise & anticlockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

  Note : Number of circular permutations of n things when p are alike and the rest are different taken all at a time distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.
- Given n different objects, the number of ways of selecting atleast one of them is
  ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$.
- Total number of ways in which it is possible to make a selection by taking some or all out of $p+q+r+\ldots\ldots$ things , where p are alike of one kind, q are alike of second kind, r are alike of third kind and so on is given by
  $[(p+1)(q+1)(r+1)\ldots\ldots\ldots] - 1$.
- Number of ways in which it is possible to make a selection of $m+n+p=N$ things, where p are alike of one kind, m are alike of second kind & n are alike of third kind taken r at a time is given by
  Coefficient of x^r in the expansion of
  $(1+x+x^2+\ldots\ldots+x^p)(1+x+x^2+\ldots\ldots+x^m) \times (1+x+x^2+\ldots\ldots+x^n)$.

REMARKS :

Remember that coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$ $(n \in N)$. For example, the number of ways in which a selection of four letters can be made from the letters of the word **PROPORTION** is given by coefficient of x^4 in $(1+x+x^2+x^3)(1+x+x^2)(1+x+x^2)(1+x)(1+x)(1+x)$.

- Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by men $= p^n$.
- Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.
- ${}^nC_r = {}^nC_{n-r}$; ${}^nC_0 = {}^nC_n = 1$;
  ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x+y=n$;
  ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_r$ is maximum if : (a) $r = \frac{n}{2}$, if n is even.
  (b) $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$, if n is odd.
- Let $N = p^a.q^b.r^c\ldots\ldots$ where $p, q, r, \ldots\ldots$ are distinct primes and $a, b, c, \ldots\ldots$ are natural numbers then
  (a) The total number of divisors of N including 1 & N is $(a+1)(b+1)(c+1)\ldots\ldots$
  (b) The sum of these divisors is
  $(p^0+p^1+p^2+\ldots.+p^a)(q^0+q^1+q^2+\ldots.+q^b)(r^0+r^1+r^2+\ldots.+r^c)\ldots.$
  (c) Number of ways in which N can be resolved as a product of two factors is
  $= \frac{1}{2}(a+1)(b+1)(c+1)\ldots$, if N is not perfect square
  [or $= \frac{1}{2}[(a+1)(b+1)(c+1)\ldots+1]$, if N is a perfect square.
  (d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1}, where n is the number of different prime factors in N.

DERANGEMENT :

If n distinct objects are arranged in a row, then the number of ways in which they can be deranged so that none of them occupies its original position is

$$n!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots\ldots+(-1)^n\frac{1}{n!}\right]$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1. What is the position of the number 321546?

(a) 267 (b) 243 (c) 271 (d) 145

2. If $f: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$, then number of onto functions satisfying $f(i) \neq i$ is equal to
(a) 44 (b) 16 (c) 9 (d) 20

3. A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw ? (All balls are distinct).
(a) 45 (b) 64 (c) 32 (d) 48

4. A five digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is
(a) 216 (b) 240 (c) 600 (d) 3125

5. Sum of all three digit numbers (no digit being zero) having the property that all digits are perfect squares, is
(a) 3108 (b) 6216 (c) 13986 (d) 2828

6. Let $a = \sum_{i<j} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j} \right)$

and $b = \sum_{i<j} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right)$, then

(a) $b = (n - 1)a$ (b) $b = (n + 1)a$
(c) $b = \frac{n}{2}a$ (d) $b = na$

7. Let $S = \{1, 2, 3, ..., n\}$. If X denote the set of all subsets of S containing exactly two distinct elements, then the value of $\sum_{A \in X} (\min A)$ is

(a) ${}^{n+1}C_3$ (b) ${}^nC_3$
(c) ${}^nC_2$ (d) None of these

8. In a certain test there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i questions where $i = 1, 2, 3, ..., n$. If the total number of wrong answers given is 2047, then n is
(a) 10 (b) 11 (c) 12 (d) 13

9. The number of ways of distribution of 10 different games among 10 children so that exactly two children get no game is
(a) ${}^{10}C_8 \frac{10! \cdot 25}{3}$ (b) ${}^{10}C_8 (10)^8$
(c) ${}^{10}C_8 \frac{10! \cdot 75}{3!}$ (d) ${}^{10}C_8 \frac{10! \cdot 25}{3!}$

10. The number of ways of arranging eight persons (having A, B, C, D, E, F among them) in a row so that A, B, C, D, E, F are always in order $ABCDEF$ (not necessarily consecutive) is
(a) ${}^8P_6$ (b) 112 (c) 28 (d) 56

11. The number of ways in which an examiner can assign 30 marks to 8 questions giving not less than 2 marks to any question, is
(a) ${}^{21}C_7$ (b) ${}^{21}C_2$ (c) ${}^{25}C_8$ (d) ${}^{30}C_8$

12. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$, then the number of ordered quadraples (a, b, c, d) is
(a) 145 (b) 165 (c) 135 (d) 155

13. If the natural numbers are written as a sequence 12345678910111213141516..... then 2007^{th} digit is
(a) 0 (b) 5 (c) 7 (d) 4

14. The number of ways of assigning 20 marks to four questions, with a maximum of 10 marks for each question is
(a) 891 (b) 981 (c) 881 (d) 791

15. Consider the set $S = \{1, 2, ..., 999, 1000\}$. D_n is the subset of S containing the elements that are divisible by n. What is the cardinality of the set $(D_2 \cup D_5) \cap D'_4$? (where D'_n denotes the complement of D_n. Cardinality of a set is defined as the number of elements contained in the set)
(a) 350 (b) 451 (c) 450 (d) 351

16. The number of ways of forming an arrangement of 4 letters from the letters of the word "IITJEE" is
(a) 66 (b) 96 (c) 102 (d) 180

17. If $n = 10800$, then the number of divisions of the form $4m + 2$, $(m \in N)$ is
(a) 10 (b) 12 (c) 14 (d) 16

18. Number of four digit numbers with all digits different and containing the digit 7 is
(a) 2016 (b) 1828 (c) 1848 (d) 1884

19. The sum of those factors of 7! which are of the form $3n + 1$ and odd $(n \in W)$ is
(a) 48 (b) 42 (c) 12 (d) 8

20. The least natural number having 12 divisors and having 21 as one of the divisors, is
(a) 126 (b) 63 (c) 84 (d) 294

21. The total number of ways in which 15 identical blankets can be distributed among four person so that each of them gets atleast two blankets is equal to

(a) ${}^{10}C_3$ (b) ${}^{9}C_3$ (c) ${}^{11}C_3$ (d) ${}^{15}C_3 \cdot {}^{7}P_4$

22. If $K = \binom{2000}{1} + \binom{2000}{2} + \binom{2000}{3} + \ldots\ldots\ldots + \binom{2000}{2000}$, then the number of digits in the value of $K + 1$ is

(a) 603 (b) 501 (c) 400 (d) 303

23. The number of ways in which the letters of the word PESSIMISTIC can be arranged so that no two S's are together, no two I's are together and letters S and I are never together is

(a) 86400 (b) 48000 (c) 2400 (d) 5480

24. If the positive integers are written in a triangle array as shown

1
2 3
4 5 6
7 8 9

Then the row in which the number of 2010 will be, is

(a) 58 (b) 61 (c) 63 (d) 65

25. The area of regular polygon of n sides with length of side $\sqrt{3}$, where $n > 1$ and satisfies the relation $\sum_{r=0}^{n} \frac{n^2 - 3n + 3}{2.{}^{n}C_r} = \sum_{r=0}^{n} \frac{r}{{}^{n}C_r}$ is

(a) $30\sqrt{3}$ (b) $\frac{3\sqrt{3}}{4}$ (c) $\frac{5\sqrt{3}}{4}$ (d) $\frac{3\sqrt{3}}{2}$

26. Rajdhani express going from Bombay to Delhi stops at five intermediate stations. 10 passengers enter the train during the journey with 10 different tickets of three classes. The number of different sets of tickets they may have is

(a) ${}^{15}C_{10}$ (b) ${}^{25}C_{10}$ (c) ${}^{45}C_{10}$ (d) ${}^{35}C_{10}$

27. The number of ways to select 6 people out of 6 married couples in which exactly one couple is present, is

(a) 60 (b) 480 (c) 660 (d) 240

28. The number of ways in which 6 different rings can be arranged on 4 fingers so that there are exactly 2 rings on one particular finger is

(a) ${}^{6}P_2 \times \frac{6!}{2!}$ (b) ${}^{6}C_2 \times \frac{6!}{2!}$

(c) ${}^{6}C_2 \times 3^4$

(d) $4^6 - 4 \times 3^6 + 60 \cdot 2^6 - 4$

29. The number of ways of arranging 15 students $A_1, A_2, \ldots, A_{15}$, in a row such that neither A_2 nor A_3 be seated before A_1 is

(a) $\frac{2\lfloor 15}{\lfloor 3}$ (b) $\frac{\lfloor 15}{\lfloor 3}$

(c) $\lfloor 2 \lfloor 15$ (d) None of these

30. The number of words that can be formed using all the letters of the word REGULATIONS such that G must come after R, L must come after A and S must come after N is

(a) $\frac{\lfloor 11}{8}$ (b) $\lfloor 11$

(c) ${}^{11}P_6$ (d) None of these

31. The number of four letter words that can be formed using the letters of the word RAMANA is

(a) 70 (b) 71 (c) 72 (d) 73

32. The number of ways in which three numbers in A.P. can be selected from 1, 2, 3,....., 21 is

(a) 55 (b) 45

(c) 100 (d) None of these

33. Before I can open my bank locker, I must remember the combination. Two of the numbers of the three-term sequence (-,-,-) are 17 and 24, but I have forgotten the third as well as the order of the numbers. The possibilities for the third number are any number between 1 and 40 (both inclusive). At 10 sec per try, at most how long will it take for me to exhaust all possibilities ? (in mins)

(a) 40 (b) 39 (c) 20 (d) 21

SECTION-II

More than One Correct Answer Type

34. Number of ways in which 200 people can be divided in 100 couples is

(a) $\frac{(200)!}{2^{100}(100)!}$ (b) $1 \cdot 3 \cdot 5. \ldots\ldots 199$

(c) $\left(\frac{101}{2}\right)\left(\frac{102}{2}\right)\ldots\ldots\ldots\left(\frac{200}{2}\right)$

(d) $\frac{(200)!}{(100)!}$

35. A forecast is to be made of the result of five cricket matches, each of which can be win or a draw or a loss for Indian team. Let

p = number of forecasts with exactly 1 going wrong

q = number of forecasts with exactly 3 going wrong

r = number of forecasts with all 5 going wrong

Then which of the following are / is true?

(a) $2q = 5r$ (b) $8p = q$

(c) $8p = 5r$ (d) $2(p + r) > q$

36. Let $S = x_1 + x_2 + x_3 + x_4 + x_5$ and $K = x_1 + x_2 + x_3$, then

(a) Number of positive integral solutions of the equation $S = 21$ is 4845.

(b) Number of integral solutions of the equation $S = 21$ such that $x_i \geq 2\ \forall i = 1, 2, 3, 4, 5$ is 1365.

(c) Number of integral solutions of the equation $S = 20$ is 42404.

(d) Number of integral solutions $(x_1, x_2, x_3, x_4, x_5)$ such that $S = 20$ and $K = 5$ is 336.

37. If $N = 8100$, then

(a) no. of positive divisors of N is 45.

(b) no. of positive divisors of N which are perfect squares is 12.

(c) no. of positive divisors of N, each of which has 4 positive divisors is 4.

(d) sum of all positive odd divisors of N is 3751.

38. If ${}^nC_{r-1} = (k^2 - 8)\cdot {}^{n+1}C_r$, then k belongs to

(a) $[-3, -2\sqrt{2})$ (b) $(-3, 3)$

(c) $(2\sqrt{2}, 3]$ (d) $[-2\sqrt{2}, 2\sqrt{2}]$

39. The number of non - negative integral solutions of $x_1 + x_2 + x_3 + x_4 \leq n$ (where n is a +ve integer) is

(a) ${}^{(n+3)}C_3$ (b) ${}^{(n+4)}C_4$

(c) ${}^{(n+5)}C_5$ (d) ${}^{(n+4)}C_n$

40. Consider $n = 21^{52}$, then

(a) number of even divisors of n is 704.

(b) number of odd divisors of n is 2809.

(c) last two digits of n is 41.

(d) number of even divisors of n which are multiple of 9 is 2705.

41. If 2050! is divisible by 17^n, then n can be

(a) 125 (b) 126 (c) 127 (d) 128

42. All the five digit numbers in which each successive digit exceeds its predecessor are arranged in the increasing order. The 105^{th} number does not contain the digit

(a) 1 (b) 3 (c) 4 (d) 5

43. 'n' locks and 'n' corresponding keys are available. But the actual combination is not known. The maximum number of trials that are needed to assign the keys to their corresponding locks are

(a) ${}^nC_2$ (b) $\sum_{k=2}^{n} (k-1)$

(c) $\sum_{k=1}^{n} k$ (d) ${}^{n+1}C_2$

44. The number of selections of 4 letters taken from the word 'COLLEGE' must be

(a) 18 (b) 22

(c) Coefficient of x^4 in the expansion of $(1 + x)^3 (1 + x + x^2)^2$

(d) Coefficient of x^4 in the expansion of $(1 + x)^2 (1 + x + x^2)^3$

45. An unlimited number of coupons bearing the letters A, B and C are available, then the number of ways of choosing 10 of these coupons so that they can't used to spell BAC

(a) $3(2^{10} - 1)$ (b) $2(3^{10} - 1)$

(c) $2^{10} - 1$ (d) 2^{10}

46. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (*i.e.* 1,16,31 etc). This process is continued until a number is reached which has already been marked, then unmarked numbers are

(a) 200 (b) 400 (c) 600 (d) 800

47. Total number of times, the digit '3' will be written, when the integers having less than 4 digits are listed, is equal to

(a) 300 (b) 310 (c) 302 (d) 306

48. The number of ways in which 5 similar pencils and 3 different pens can be distributed to 7 children where each child gets at least one object is

(a) 840 (b) 630 (c) 1596 (d) 1470

SECTION-III

Comprehension Type

Paragraph for Question No. 49 to 51

10 digit numbers are formed by using all the digits 0, 1, 2,...., 9 such that they are divisible by 11111.

Then, answer the following questions.

49. The digit in tens place, in smallest of such numbers is

(a) 9 (b) 8 (c) 7 (d) 6

50. The digit in units place, in greatest of such numbers is

(a) 4 (b) 3 (c) 2 (d) 1

51. The total number of such numbers is

(a) 6543 (b) 5634 (c) 3456 (d) 4365

Paragraph for Question No. 52 and 53

There are three cards in a bag, each card having a different natural number. The three players A, B, C play a game such that, in each round the cards are drawn from the bag, in order by A, B and C without replacement (after completion of every round the cards are put back in the bag). After two or more rounds, player B has sum total of numbers drawn 20, player A has sum total of numbers drawn 10 and player C has sum total of numbers 9. In the last round the player A received the largest number.

52. The greatest number is

(a) 7 (b) 8 (c) 6 (d) 5

53. Total number of rounds played is

(a) 5 (b) 4 (c) 3 (d) 2

Paragraph for Question No. 54 to 56

Let $x_1, x_2, \ldots\ldots, x_n$ be n integers. Then no. of solutions to the equation $x_1 + x_2 + \ldots + x_m = n$ subject to the condition $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2, \ldots\ldots, a_m \le x_m \le b_m$ is equal to the coefficient of x^n in $(x^{a_1} + x^{a_1+1} + \ldots\ldots + x^{b_1})(x^{a_2} + x^{a_2+1} + \ldots\ldots + x^{b_2})(x^{a_m} + x^{a_m+1} + \ldots\ldots + x^{b_m})$. The coefficient of x^r in the expansion of $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$.

54. If $x, y, z \in N$ and r, s, t be distinct prime numbers, then the number of positive integral solutions of the equation $xyz = r^2s^3t^2$ are

(a) 231 (b) 241

(c) 360 (d) none of these

55. The number of ways of selecting n things out of $3n$ things of which n are of one kind and alike and n are of second kind and alike and rest are unlike is

(a) $(n + 1)^2 2^n$ (b) $(n + 2)2^{n-1}$

(c) $(n + 1)^2 2^n - 1$ (d) none of these

56. The number of ways so that we can put n distinct balls into r distinct boxes if empty box is not allowed is equal to

(a) total number of functions from set A to set B having cardinal numbers $n(A) = n$ and $n(B) = r$.

(b) total number of onto functions from set A to set B having cardinal numbers $n(A) = n$ and $n(B) = r$

(c) ${}^{n-1}C_{r-1}\, n!$ (d) ${}^{n-1}C_{r-1}$

Paragraph for Question No. 57 to 59

There are 18 points in a plane out of which 3 of them fall on a line L_{10}, another 4 of them fall on another line $L_2 = 0$, another five lie on a different line $L_3 = 0$. No three points other than those stated above are collinear.

57. Number of lines formed by joining the points on the plane is

(a) 134 (b) 137 (c) 124 (d) 130

58. Number of triangles formed by joining the points on the plane is

(a) 801 (b) 800

(c) 821 (d) None of these

59. If the six points that do not lie on $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are now removed. Find the number of n-gon's (polygons with n sides) where $n > 3$ such that not more than two points are selected from each line.

(a) 848 (b) 940

(c) 948 (d) None of these

Paragraph for Question No. 60 to 62

Four prizes are distributed among 5 students

60. If no student gets more than one prize, then the number of ways is

(a) 5^4 (b) ${}^5P_4$ (c) 4^5 (d) 620

61. If each student is eligible for all the prizes, then the number of ways is

(a) 625 (b) 620 (c) 1024 (d) 1020

62. If no student gets all the prizes, then the number of ways is

(a) 625 (b) 620 (c) 1024 (d) 1020

Paragraph for Question No. 63 to 65

Let $x_1x_2x_3x_4x_5x_6$ be a six digit number, then find the number of such numbers such that

63. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ is

(a) ${}^9C_6$ (b) ${}^{10}C_3$ (c) ${}^{10}C_6$ (d) ${}^{10}C_5$

64. $x_1 < x_2 < x_3 = x_4 < x_5 < x_6$ is

(a) ${}^9C_3$ (b) ${}^9C_4$ (c) ${}^{10}C_3$ (d) ${}^{10}C_4$

65. $x_1 < x_2 < x_3 \leq x_4 < x_5 < x_6$ is

(a) ${}^9C_4$ (b) ${}^9C_3$ (c) ${}^{10}C_4$ (d) ${}^{10}C_3$

Paragraph for Question No. 66 to 68

Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if

66. Balls and boxes are all different

(a) 50 (b) 100 (c) 125 (d) 150

67. Balls are identical but boxes are different

(a) 2 (b) 6 (c) 4 (d) 8

68. Balls are different but boxes are identical

(a) 25 (b) 15

(c) 10 (d) 35

SECTION-IV

Matrix-Match Type

69. A set A has 6 elements. Then the number of bijections 'f' from A to A such that

Column-I		Column-II	
(A)	$f(n) \neq n$ for any $n \in A$ is	(p)	40
(B)	$f(n) = n$ exactly for 3 elements is	(q)	36
(C)	$f(n) = n$ for atleast 3 elements is	(r)	265
(D)	odd integers are mapped to odd integers and even integers are mapped to even integers is	(s)	56

SECTION-V

Integer Answer Type

70. Number of four digit numbers formed by the digits 1, 2, 3, 4, 5, 6, 7 which are divisible by 25 is $8m$, then m =

71. There are four balls of different colours and four boxes of the same colours as those of the balls. The number of ways in which the balls could be placed one each in a box, such that no ball goes into the box of its own colour, is

72. If the number of different words that can be formed using all the letters of the word "MAHAVEER" where two of the vowels are together and the other two are also together but separated from the first two is S, then $S - 1432$ =

73. No. of positive integers 'n' such that $n!$ contains six zeroes is

74. The ten's digit of $1! + 2! + 3! + \ldots\ldots + 1000!$ is

75. Consider the 12 face diagonals of a rectangular block. The number of pairs of skew lines are $10l$, where l =

76. The number of acute triangles whose vertices are chosen from the vertices of a rectangular block are

77. There are two die A and B both having six faces. Die A has three faces marked with 1, two faces with 2, and one face marked with 3. Die B has one face marked with 1, two faces marked with 2 and three faces marked with 3. Both dies are thrown randomly once. If E be the event of getting sum of the numbers appearing on top faces equal to x. $P(E)$ is maximum if x is

SOLUTIONS

1. (a) : $N = 321546$

Number of numbers beginning with 1 = 120

1					

Number of numbers beginning with 2 = 120

2					

Number of numbers starting with 31= 24

3	1				

starting with 3214 = 2

3	2	1	4		

finally 321546 = 1

3	2	1	5	4	6

Hence N has 267^{th} position.

2. (a) : Number of onto functions

$$= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

3. (b) : There are 9 balls in the box out of which three balls can be drawn in ${}^9C_3$ ways. But these ways also include those ways in which 3 black balls

are drawn. Total number of ways when black ball is not drawn is ${}^6C_3$.
Hence, the number of ways in which at least one black ball is included = Total number of ways of drawing 3 balls from 9 balls – Number of ways of drawing 3 balls from 6 balls = ${}^9C_3 - {}^6C_3 = 84 - 20 = 64$

4. (a) : Use 1,2,3,4,5, then number of ways = 5!
Fix '0' at unit place, tenth place, hundredth place and thousandth place then number of ways = $4 \times 4!$
$\therefore$ Total number of ways = 216

5. (c) : Use 1, 4, 9
Sum = $9(14) \times 111 = 126 \times 111 = 13986$

6. (c) :

$$2b = \sum_{i<j}\left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right) + \sum_{n-j<n-i} \frac{n-j}{{}^nC_j} + \frac{n-i}{{}^nC_i}$$

$$= \sum_{i<j} \frac{n}{{}^nC_i} + \frac{n}{{}^nC_j} = n \cdot a \Rightarrow b = \frac{n}{2}a$$

7. (a) : Let $S = \{1,2,3,.....,n\}$ each number i is minimum of $(n - i)$ sets.

$$\therefore \sum_{A \in X} \min A = \sum_{i=1}^{n} i(n-i)$$

$$= (n)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = \frac{(n)n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{6}(3n-2n-1) = \frac{n(n+1)(n-1)}{6} = {}^{n+1}C_3$$

8. (b) : Number of students who gave wrong answers to atleast i questions = 2^{n-i}
Number of students who gave wrong answers to exactly n questions = No. of students who gave wrong answers to atleast n questions = $2^0 = 1$
Number of students gave wrong answers to exactly i questions = $2^{n-i} - 2^{n-(i+1)}$
Number of wrong answers

$$= \sum_{i=1}^{n-1} i\left(2^{n-i} - 2^{n-(i+1)}\right) + (n) = 2047$$

$\Rightarrow 1 + 2^1 + 2^2 + ... + 2^{n-1} = 2047$
$\Rightarrow 2^n = 2048 \Rightarrow n = 11$

9. (a) : Required number of ways

$$= {}^{10}C_8\left[{}^8C_1\frac{10!}{3!} + {}^8C_2\frac{10!}{2!2!}\right] = {}^{10}C_8 \cdot \frac{10! \cdot 25}{3}$$

10. (d) : ${}^8P_{8-6} = {}^8P_2 = 56$

11. (d) : Coefficient of x^{30} in $(x^2 + x^3 + x^4 + ... + x^{16})^8$
$= {}^{21}C_{14} = {}^{21}C_7$

12. (b) : $(2k_1+1)+(2k_2+1)+(2k_3+1)+(2k_4+1) = 20$
$\Rightarrow k_1 + k_2 + k_3 + k_4 = 8$
where $k_i \geq 0$, $i = 1, 2, 3, 4$
$\therefore$ Number of ordered quadraples
$= {}^{(4+8-1)}C_8 = {}^{11}C_3 = 165$

13. (b) : Single digited = 9
Two digited = $9 \times 10 = 90 \rightarrow$ positions = 180
Three digited start with 1 = 100 $\rightarrow$ positions = $100 \times 3 = 300$
Three digited start with 2 $\rightarrow$ positions = 300
Three digited start with 3 $\rightarrow$ positions = 300
Three digited start with 4 $\rightarrow$ positions = 300
Three digited start with 5 $\rightarrow$ positions = 300
Three digited start with 6 $\rightarrow$ positions = 300

700 $\rightarrow$ 3
701 $\rightarrow$ 3
702 $\rightarrow$ 3
703 $\rightarrow$ 3
704 $\rightarrow$ 3
705 $\rightarrow$ 3

Total positions = 2007
$\therefore$ 2007^{th} digit is '5'.

14. (a) : Required number of ways = Coefficient of x^{20} in $(1 + x + x^2 + ... + x^{10})^4$
$= {}^{23}C_3 - 4({}^{12}C_3) = 891$

15. (a) : Let # A denote the cardinality of a finite set A.
$\#\, D_2 = 500$, $\#\, D_5 = 200$, $\#\,(D_2 \cap D_5) = 100 = \#(D_{10})$
[$\because$ L.C.M of (2,5) = 10]
Since $\#\,(D_2 \cup D_5) = \#D_2 + \#D_5 - \#\,(D_2 \cap D_5)$
$\Rightarrow \#\,(D_2 \cup D_5) = 600$ and $\#\, D_4 = 250$
$\#\,(D_2 \cup D_5) \cap D_4') = \#((D_2 \cup D_5) - D_4) = 600 - 250 = 350$

16. (c) : Number of arrangements of 4 letters all of which are different = 4!
Number of arrangements of 4 letters consisting of 2 identical and 2 different $= {}^2C_1 \times {}^3C_2 \times \frac{4!}{2!}$
Number of arrangements of 4 letters consisting of 2 identical of one kind and 2 identical of another kind $= \frac{4!}{(2!)^2}$
Sum = 24 + 72 + 6 = 102

17. (b) : $4m + 2 = 2(2m + 1)$

2 must occur exactly once

$1 \times (3 + 1)(2 + 1) = 12$

18. (c) : Excluding 7, remaining 3 digits can be selected in ${}^9C_3$ ways and arranged in 4! ways.

Number of ways = ${}^9C_3 \cdot 4!$ – (when '0' and '7' are always included and '0' occupies the first place)

$= {}^9C_3 \cdot 4! - {}^8C_2 \cdot 3! = 84 \cdot 4 \cdot 3! - 28 \cdot 3!$

$= 7 \cdot 3![48 - 4] = 6 \cdot 7 \cdot 44 = 1848$

19. (d) : $7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, factors of form $3n + 1$ and odd only, have sum 8.

20. (c) : Let $n = 2^a \times 3^b \times 5^c \times .....$

For least natural number and 21 as one factor

$n = 2^2 \times 3^1 \times 7^1 = 84$

21. (a) : $x_1 + x_2 + x_3 + x_4 = 15,\ x_i \geq 2$

or $y_1 + y_2 + y_3 + y_4 = 7;\ y_i = x_i - 2 \geq 0$

$\therefore$ Total number of ways $= {}^{7+4-1}C_{4-1} = {}^{10}C_3$

22. (a) : $f(x) = (1+x)^{2000} = \sum_{K=0}^{2000} \binom{2000}{K}$

$\therefore \quad K = 2^{2000} - 1$

$\Rightarrow \quad K + 1 = 2^{2000} \Rightarrow \log_{10}(K+1) = 2000 \times 0.3010$

$= 602$

$\therefore$ Number of digits = 603

23. (c) : Arrange the remaining letters in 5! ways

$\left|P\right|E\left|M\right|T\left|C\right|$

In the 6 gaps arrange the 3 Ss and 3 Is

$$\text{in } \frac{6!}{3!\,3!} = 20 \text{ ways}$$

$\therefore$ The answer is $5! \times 20 = 2400$.

24. (c) : Let 2010 be in k^{th} row

$\Rightarrow$ k^{th} term of series 1, 2, 4, 7,..... < 2010

and $(k+1)^{\text{th}}$ term of series 1, 2, 4, 7,.... > 2010

$S_n = 1 + 2 + 4 + 7 + ... + T_n$

$S_n = \quad 1 + 2 + 4 + ... + T_{n-1} + T_n$

$\Rightarrow \quad 0 = 1 + (1 + 2 + 3 + ... (n-1) \text{ terms}) - T_n$

$$\Rightarrow \quad T_n = \frac{n^2 - n + 2}{2}$$

$$\Rightarrow \quad \frac{k^2 - k + 2}{2} \leq 2010 \text{ and } \frac{k^2 + k + 2}{2} > 2010$$

$\Rightarrow \quad k^2 - k - 4018 \leq 0$ and $k^2 + k - 4018 > 0$

$$\Rightarrow \quad \left(k - \frac{1}{2}\right)^2 \leq \frac{16073}{4} \text{ and } \left(k + \frac{1}{2}\right)^2 > \frac{16073}{4}$$

$$\Rightarrow \quad k - \frac{1}{2} \leq 63.3 \text{ and } k + \frac{1}{2} > 63.3$$

$\Rightarrow \quad k = 63$

25. (b) : $\sum \frac{r}{{}^nC_r} = \sum \frac{n-r}{{}^nC_r} = \frac{n}{2} \sum \frac{1}{{}^nC_r}$

$$= \frac{n^2 - 3n + 3}{2} \sum \frac{1}{{}^nC_r} \Rightarrow n^2 - 3n + 3 = n$$

$\Rightarrow \quad n^2 - 4n + 3 = 0 \quad \Rightarrow \quad n = 3 \ (\because \ n > 1)$

$\therefore$ Area of equilateral triangle with side $\sqrt{3}$ is $\frac{3\sqrt{3}}{4}$.

26. (c) : $B, s_1, s_2, s_3, s_4, s_5, D$

For a particular class the total number of different tickets from first intermediate station is 5.

Number of different tickets from second intermediate station is 4.

So the total no. of different tickets is 5 + 4 + 3 + 2 +1 = 15

For the remaining classes the total no. of different tickets = $2 \times 15 = 30$

$\therefore$ Total number of different tickets for all three classes = $3 \times 15 = 45$

Required number of ways = ${}^{45}C_{10}$

27. (b) : Required number of ways

$= {}^6C_1 \times {}^5C_4 \times 2^4 = 480$

28. (a) : Arrange two rings on one particular finger in ${}^6P_2$ ways.

Now, the remaining 4 rings can be arranged in remaining 3 fingers in $3 \times 4 \times 5 \times 6 = \frac{6!}{2!}$ ways.

The answer is ${}^6P_2 \times \frac{6!}{2!}$.

29. (a) : First arrange 12 persons $A_4, A_5, ..., A_{15}$ in ${}^{15}P_{12}$ ways. There remains 3 places. Keep A_1 in the first place and arrange A_2, A_3 in the remaining two places in 2! ways

$\therefore$ Required number of arrangements = ${}^{15}P_{12} \cdot 1 \cdot 2!$

30. (a) : The other 5 letters in 11 places can be arranged in ${}^{11}P_5$ ways. Then there remains 6 places. Choose two of them in ${}^6C_2$ ways, in these places we can put G, R only in one way. Similarly for others.

$\therefore$ Required no. of ways $= {}^{11}P_5 \cdot {}^6C_2 \cdot {}^4C_2 \cdot {}^2C_2 = \frac{\lfloor 11}{8}$

31. (c) : Four letter words in which all the letters are different $= \lfloor\underline{4} = 24$

Four letter words in which 3 letters are alike and one is different $= \frac{\lfloor\underline{4}}{\lfloor\underline{3}} \times 3 = 12$

Four letter words in which two letters are alike and rest are different $= \frac{4!}{2!} \times 3 = 36$

$\therefore$ Required number of ways $= 24 + 12 + 36 = 72$

32. (c) : If a, b, c are in A.P., then $2b = a + c$

$\Rightarrow$ $a + c$ is even.

$\therefore$ Both a & c are either even or odd.

From 1 to 21 there are 10 even and 11 Odd numbers.

$\therefore$ Required number of ways $= {}^{10}C_2 + {}^{11}C_2 = 100$

33. (b) : Consider the six subsets of possible combinations.

Let $A_1 = \{(x, 17, 24) | 1 \le x \le 40\}$

$A_2 = \{(x, 24, 17) | 1 \le x \le 40\}$

$A_3 = \{(17, x, 24) | 1 \le x \le 40\}$

$A_4 = \{(24, x, 17) | 1 \le x \le 40\}$

$A_5 = \{(17, 24, x) | 1 \le x \le 40\}$

$A_6 = \{(24, 17, x) | 1 \le x \le 40\}$

In each set, there are 40 elements but some combinations have been counted twice, like (17, 17, 24) belongs to both A_1 and A_3. And in total there are 6 such elements.

So, net possibilities $= 240 - 6 = 234$

Hence time taken is 39 mins.

34. (a, b, c) : $\frac{(200)!}{\underbrace{2!.2!.....2!}_{100 \text{ times}}(100)!} = \frac{(200)!}{100! \cdot 2^{100}}$

$= 1 \cdot 3 \cdot 5 \ldots\ldots\ldots \cdot 199$

$\frac{(200)!}{100! \cdot 2^{100}} = \left(\frac{101}{2}\right)\left(\frac{102}{2}\right)\ldots\ldots\ldots\ldots\ldots\left(\frac{200}{2}\right)$

35. (a, b, d) : $p = {}^5C_4 \times {}^2C_1 = 10$;

$q = {}^5C_2 \times ({}^2C_1)^3 = 80$; $r = {}^5C_0 \times ({}^2C_1)^5 = 32$

$2q = 5r$, $8p = q$ and $2(p + r) > q$

36. (a, b, d) : (a) ${}^{21-1}C_{5-1} = 4845$

(b) coefficient of x^{21} in $(x^2 + x^3 + \ldots\ldots + x^{13})^5$

$= {}^{15}C_{11} = 1365$

(c) ${}^{24}C_4 = 10626$

(d) ${}^{16}C_1 \times {}^7C_2 = 336$

37. (a, b, c, d) : $N = 2^2 3^4 5^2$

(a) No. of positive divisors $= (2 + 1)(4 + 1)(2 + 1) = 45$

(b) No. of positive divisors $= (1 + 1)(2 + 1)(1 + 1) = 12$

(c) No. of positive divisors $= 1 + 3 = 4$

(d) Sum of all positive odd divisors

$= \left(\frac{3^5 - 1}{3 - 1}\right)\left(\frac{5^3 - 1}{5 - 1}\right) = 3751$

38. (a, c) : $r - 1 \ge 0, r \le n + 1 \Rightarrow 1 \le r \le n + 1$

$\Rightarrow \frac{1}{n+1} \le \frac{r}{n+1} \le 1$

Also $k^2 - 8 = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} = \frac{r}{n+1}$

$\therefore \frac{1}{n+1} \le k^2 - 8 \le 1 \Rightarrow 8 < k^2 \le 9$

$\Rightarrow k \in [-3, -2\sqrt{2})$ or $k \in (2\sqrt{2}, 3]$

39. (b, d) : Let x_5 be such that $x_1 + x_2 + x_3 + x_4 + x_5 = n$

$\therefore$ No. of non-negative integral solutions

$= {}^{n+5-1}C_{5-1} = {}^{n+4}C_4$

40. (b, c) : $n = 3^{52}7^{52}$

So number of odd divisors $= 53 \times 53 = 2809$

$21^{52} = (20 + 1)^{52} = {}^{52}C_0 20^{52} + {}^{52}C_1 20^{51} + \ldots\ldots + {}^{52}C_{51} 20 + 1$

$= 100k' + (52 \times 20) + 1 = 100k + 41$

Number of divisors which are multiple of $9 = 53 \times 51$

$= 2703.$

41. (a, b, c) : $E_{17}(2050!) = \left[\frac{2050}{17}\right] + \left[\frac{2050}{17^2}\right] + \ldots$

$= 120 + 7 + 0 + 0 + \ldots$

So in 2050! exponent of prime 17 are 127. So n can be 125, 126 and 127. So answers are (a, b, c)

42. (a, b, c, d) : The five digit numbers starting with 1 are ${}^8C_4 = 70$ | 1 | | | | |

The five digit numbers starting with 2 are

${}^7C_4 = 35$ | 2 | | | | |

Total $= 70 + 35 = 105$

$\therefore$ $(105)^{\text{th}}$ five digit no. is 26789

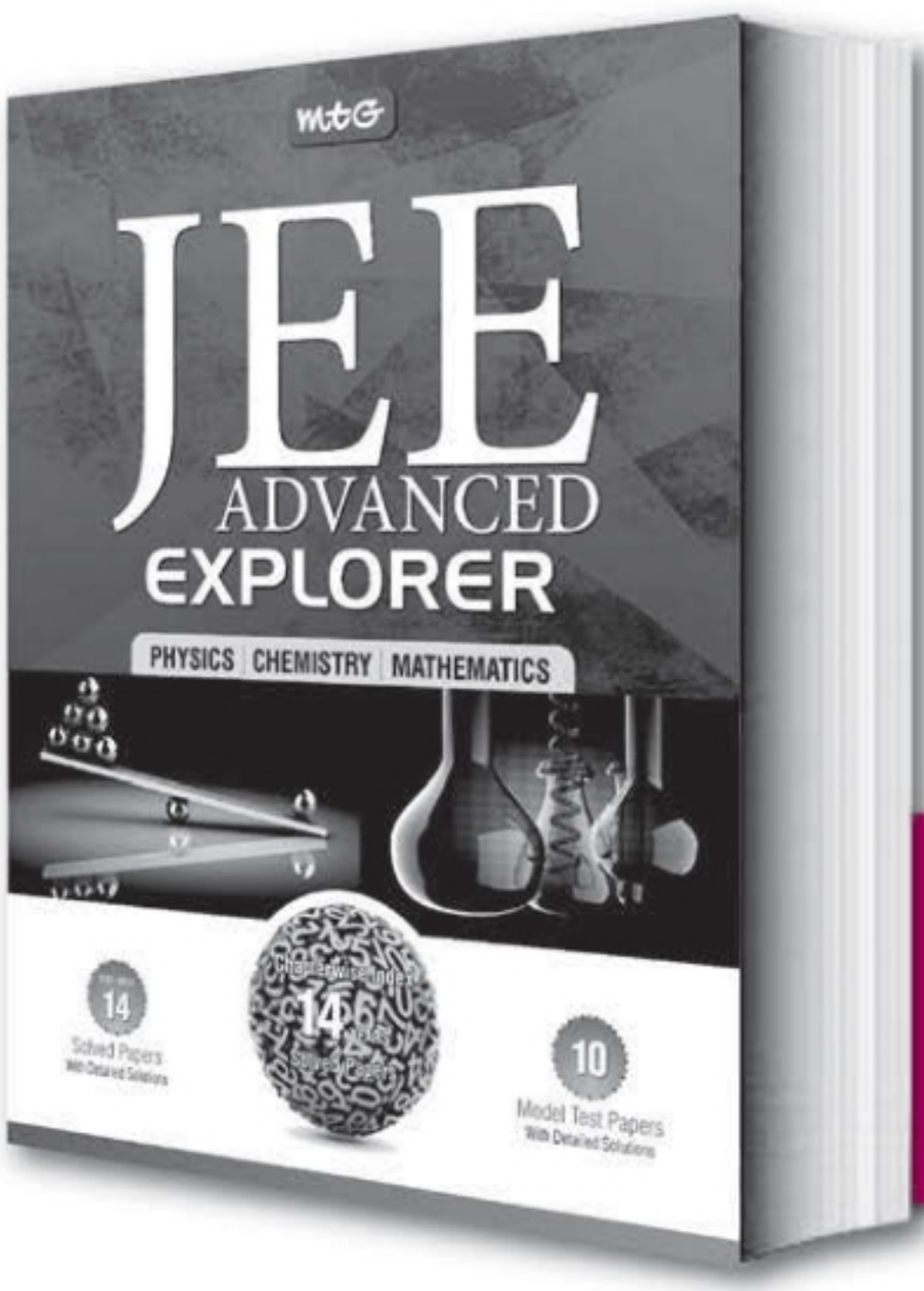
43. (c, d) : The maximum number of trials required

$= (n) + (n - 1) + (n - 2) + \ldots\ldots + 2 + 1$

$= \frac{n(n+1)}{2} = {}^{n+1}C_2 = \sum_{k=1}^{n} k$

JEE (ADVANCED)

Dry runs are here!



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44. (a, c) : The required number of selections of 4 letters = coeff. of x^4 in the expansion of $(x^0 + x^1)(x^0 + x^1)(x^0 + x^1 + x^2)(x^0 + x^1 + x^2)(x^0 + x^1)$ = coeff. of x^4 in $(1 + x)^3(1 + x + x^2)^2 = 18$

45. (a) : Case I :
When all the selected coupons bear the same letter. One letter can be selected from three letters in ${}^3C_1$ ways.
$\Rightarrow$ Total number of ways of choosing 10 coupons bearing the same letter is ${}^3C_1 \times 1 = 3$
Case II :
When selected coupons bear two letters only
The number of ways of selecting two letters from 3 is ${}^3C_2$
$\Rightarrow$ The number of ways in which selected coupons bear two letters only = ${}^3C_2(2^{10} - 2)$
Hence required number of ways = $3 + {}^3C_2(2^{10} - 2)$
$= 3(2^{10} - 1)$

46. (a, b, c, d) : In 1st round all the integers, which leaves the remainder 1 when divided by 15, will be marked Last number of this category is 991
Next number to be marked is (991+15 – 1000) = 6
Again, second round of integers which leaves the remainder '6' when divided by 15 will be marked.
Last number of this category is 996.
Next number to be marked is (996+15-1000) = 11
Thus third round of integers which leaves the remainder 11 when divided by 15, will be marked .
Last number of this category is 986.
Next number to be marked is 986 + 15 – 1000 = 1 which is already been marked.

47. (a) : The number of times the digit 3 will be written in the unit place is 100. Similarly in the tens place & hundred place.
$\therefore$ Ans is 300

48. (c) : Divide 8 objects in 7 packs and distribute them.

$$1 \times \frac{7!}{3!} + {}^3C_1 \times \frac{7!}{4!} + {}^3C_2 \times \frac{7!}{5!} = 1596$$

49. (d) 50. (a) 51. (c)
Let $N = x_1x_2x_3 \ldots\ldots x_{10}$ be one of such numbers
$\because \sum x_i = 45$, hence 'N' is divisible by 9, but 11111 is not divisible by 9. Hence N should be divisible by $11111 \times 9 = 99999$
Now $x_1x_2x_3 \ldots\ldots x_{10} = x_1x_2x_3x_4x_5 \times 10^5 + x_6x_7x_8x_9x_{10}$
$= x_1x_2x_3x_4x_5 \times 99999 + (x_1x_2x_3x_4x_5) + (x_6x_7x_8x_9x_{10})$
Now $x_1x_2x_3x_4x_5 < 99999$ and $x_6x_7x_8x_9x_{10} < 99999$
$\therefore \quad x_1x_2x_3x_4x_5 + x_6x_7x_8x_9x_{10} < (2 \times 99999)$
$\Rightarrow \quad (x_1x_2x_3x_4x_5 + x_6x_7x_8x_9x_{10}) = 99999$
$\Rightarrow \quad x_1 + x_6 = x_2 + x_7 = x_3 + x_8 = x_4 + x_9 = x_5 + x_{10} = 9$
d – Smallest number = 1023489765
a – Biggest number = 9876501234
c – Total ways = $9 \times 8 \times 6 \times 4 \times 2 \times 1 \times 1 \times 1 \times 1 \times 1 = 3456$

52. (b) 53. (c)
Let the numbers be x_1, x_2, x_3 and sum total of numbers after n rounds is
$n(x_1 + x_2 + x_3) = 39$, where $n \geq 2$ and $x_i \leq 9$ and factors of 39 is 1,3,13, 39 hence $n = 3$.
$\Rightarrow \quad x_1 + x_2 + x_3 = 13$
Since sum total of B after 3 rounds is 20 and that of A is 10.
Let $x_1 < x_2 < x_3 \Rightarrow x_3 \geq 7$ and $x_3 \leq 8$
Hence possible values are $x_3 = 8$, $x_2 = 4$, $x_1 = 1$.

54. (c): ${}^4C_2 \cdot {}^5C_2 \cdot {}^4C_2 = 360$

55. (b) : Coefficient of x^n in
$(1 + x + x^2 + \ldots\ldots + x^n)^2(1 + x)^n$
= coefficient of x^n in $(1 - x)^{-2}(1 + x)^n = (n + 2)2^{n-1}$

56. (b)

57. (b) : Number of lines formed by joining the points = ${}^{18}C_2 - ({}^3C_2 + {}^4C_2 + {}^5C_2) + 3 = 137$

58. (a) : Number of triangles formed = ${}^{18}C_3 - ({}^3C_3 + {}^4C_3 + {}^5C_3) = 801$

59. (c) : Number of quadrilaterals
$= {}^4C_2 \times {}^5C_2 + {}^3C_2 \times {}^5C_2 + {}^3C_2 \times {}^4C_2 + {}^3C_1 \times {}^4C_2 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^5C_2 + {}^3C_2 \times {}^4C_1 \times {}^5C_1 = 378$
Number of pentagons
$= {}^3C_1 \times {}^4C_2 \times {}^5C_2 + {}^3C_2 \times {}^4C_1 \times {}^5C_2 + {}^3C_2 \times {}^4C_2 \times {}^5C_1 = 390$
$\therefore$ Total number of n – gons = 948

60. (b) : Required number of ways = ${}^5P_4$

61. (a) : Each prize can be given in 5 ways
$\therefore$ Required number of ways = $5 \cdot 5 \cdot 5 \cdot 5 = 625$

62. (b) : All prizes can be given to one student in 5 ways
$\Rightarrow$ Required number of ways = $5^4 - 5$

63. (a) : $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \rightarrow {}^9C_6$ ways

64. (b) : $x_1 < x_2 < x_3 = x_4 < x_5 < x_6 \rightarrow {}^9C_4$ ways

65. (c) : $x_1 < x_2 < x_3 \leq x_4 < x_5 < x_6 \rightarrow {}^9C_6 + {}^9C_4 = {}^9C_3 + {}^9C_4 = {}^{10}C_4$ ways

66. (d) : Balls =5, Box = 3
Balls & boxes are all different.
Req. no. of ways $= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 - {}^3C_3(3-3)^5$

$= 243 - 3(32) + 3(1) - 0$
$= 243 - 96 + 3 = 243 - 93 = 150$

67. (b) : $x_1 + x_2 + x_3 = 5, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$
$k + l + m = 2, k \geq 0, l \geq 0, m \geq 0$
$\therefore$ Required number of ways $= {}^{n+r-1}C_{r-1} = {}^4C_2 = 6$

68. (a) : The balls should be divided into two groups as

$$\frac{5!}{1!3!2!} + \frac{5!}{1!2!2!2!} = 10 + 15 = 25$$

69. (A) → (r); (B) → (p); (C) → (s); (D) → (q)

(A) $6!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$

$= 360 - 120 + 30 - 6 + 1 = 265$

(B) ${}^6C_3 \times 3!\left(\frac{1}{2!} - \frac{1}{3!}\right) = 20 \times 2 = 40$

(C) ${}^6C_3 \times 2 + {}^6C_4 \times 1 + {}^6C_6 = 40 + 15 + 1 = 56$

(D) $3! \times 3! = 36$

70. (5) Last two digits are 25, 75 remaining places can be filled in ${}^5P_2$ ways. Hence four digit numbers formed are $= 40 = 8m \implies m = 5$

71. (9) $4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$ (number of derangements)

72. (8) (EE) ↓ M ↓ H (AA) ↓ R ↓ V ↓

M, H, R, V can be arranged in 4! ways.
Now, there are 5 places where the 2 packages of 2 vowels each can be put.
$\therefore$ Number of ways $= {}^5C_2$

Case I :

When (EE) and (AA) are together

Number of ways $= {}^5C_2 \times 2! \times 4!$((EE) and (AA) can interchange their places)

Case II :

When (EA) and (EA) are together

Number of ways $= {}^5C_2 \times 4! \times 2! \times 2!$

$\therefore$ Total number of ways $= {}^5C_2 \times 4! \times 2!(1+2)$

$$= \frac{5!}{2!3!} \times 4! \times 2! \times 3 = \frac{5! \times 4!}{2!}$$

$= 120 \times 12 = 1440$

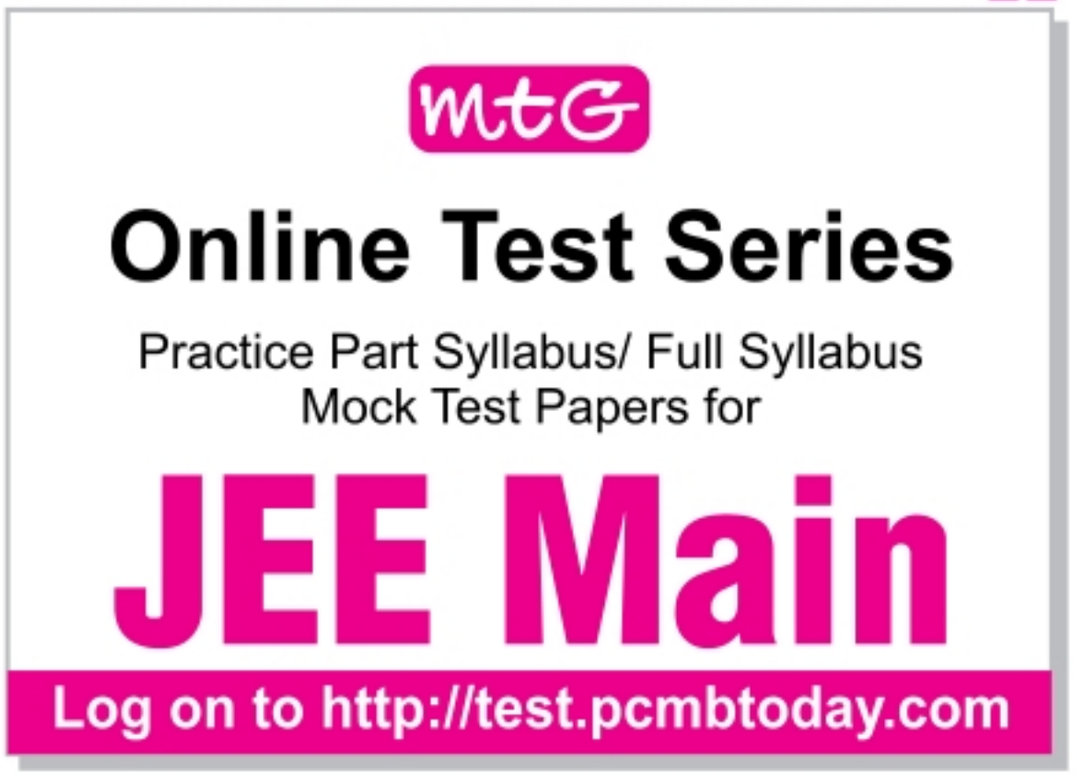
73. (5) : Exponent of 5 in 25! is 6
Then exponent of 5 in 30! is 7
$\therefore$ 25!, 26!, 27!, 28!, 29!

74. (1) : $1! + 2! + 3! + 4! = 33$
$5! = 120, 6! = 720, 7! = 5040, 8! = 40320,$
$9! = 362880$
After 9! the other numbers will be divisible by 100 hence won't make any contribution in tens place.

75. (3) Consider a fixed face diagonal. It is parallel to the corresponding one on the opposite face. It intersects the other diagonal of the same face, and one diagonal on each of the four neighbouring faces. The other five face diagonals form 5 pairs of skew lines with the fixed one. By symmetry,
total $= 12 \times 5 = 60$ but since each pair is counted twice. Hence, net skew pairs = 30

76. (8) The 8 vertices of the block gives ${}^8C_3 = 56$ triangles, each of which is either acute or right angled. Each vertex of the block serves as the vertex of the right angle in three triangles whose hypotenuse are face diagonals of the block, and three triangles whose hypotenuse are space diagonals of the block. Hence there are $8(3+3) = 48$ right triangles and so $56 - 48 = 8$ acute triangles.

77. (4) x can be 2, 3, 4, 5, 6. The number of ways in which sum of 2, 3, 4, 5, 6 can occur is given by the coefficients of x^2, x^3, x^4, x^5, x^6 in $(3x + 2x^2 + x^3) \times (x + 2x^2 + 3x^3) = 3x^2 + 8x^3 + 14x^4 + 8x^5 + 3x^6$
This shows that sum that occurs most often is 4.



concept BOOSTERS

Class XII

Probability

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main / Advanced and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

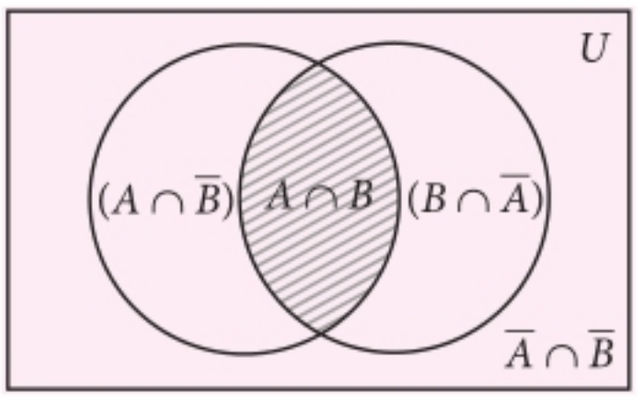
GALLERY OF RESULTS

- **Compound event :** An event consisting of two or more independent events is called compound event. If A and B are any two events then probability of compound event is denoted and defined by;
  $P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$
- **Classical definition of probability :** If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by $P(A) = m/n$.
- **Sample space :** The set of all possible outcomes of an experiment is called the sample space (S).
- **Event :** A subset of a sample space is called an event.
- **Complement of an event A :** The set of all outcomes which are in S but not in A is called the complement of the event A and denoted by $\bar{A}$ or A^c.
- **Exhaustive events :** Events $A, B, C, \ldots\ldots, L$ are said to be exhaustive events if no event outside this set can result as an outcome of an experiment. Let A and B be any two events defined on a sample space S, then A and B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.
- **Mutually exclusive events :** Two events are said to be mutually exclusive (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other . If A and B are two mutually exclusive events, then $P(A \cap B) = 0$.
- **Equally likely events :** Events are said to be equally likely when each event, is as likely to occur as any other event.

REMARKS :

- If x cases are favourable to A and y cases are favourable to $\bar{A}$, then $P(A) = \dfrac{x}{(x+y)}$ and $P(\bar{A}) = \dfrac{y}{(x+y)}$. We say that odds in favour of A are $x : y$ and odds against of A are $y : x$.
- $P(A) + P(\bar{A}) = 1$, where $\bar{A} = \text{not } A$
- $0 \le P(A) \le 1$

  $A \cup B = A + B = A$ or B denotes occurrence of at least A or B. For 2 events A and B :



- If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
- If A and B are any two events, then $P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$,

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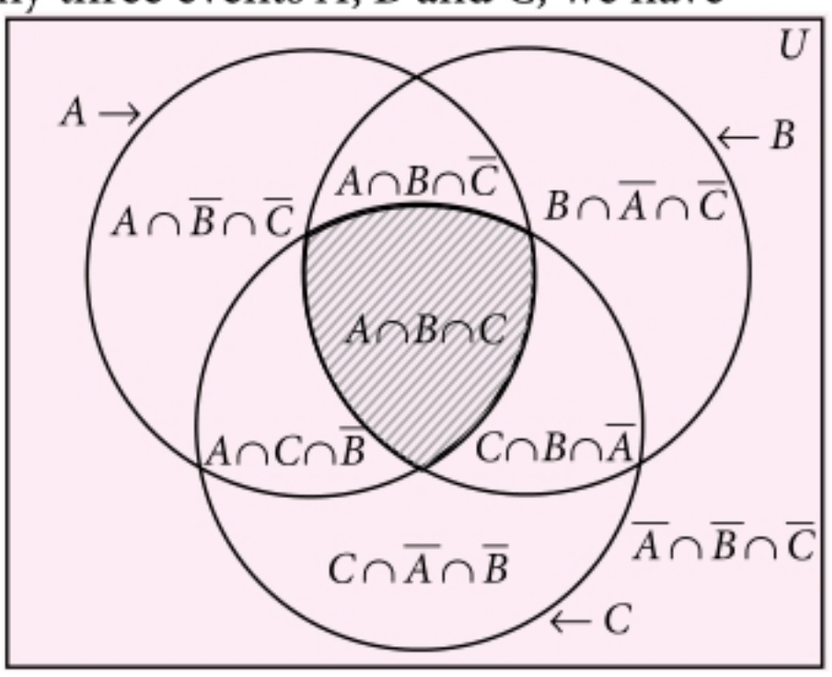
where $P(B/A)$ means conditional probability of B given A and $P(A/B)$ means conditional probability of A given B.

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  $= P(A \cdot \overline{B}) + P(\overline{A} \cdot B) + P(A \cdot B) = 1 - P(\overline{A} \cdot \overline{B})$
- Opposite of "atleast A or B" is neither A nor B
  i.e. $P(\overline{A \cup B}) = 1 - P(A \text{ or } B) = P(\overline{A} \cap \overline{B})$
  $\Rightarrow P(A \cup B) + P(\overline{A} \cap \overline{B}) = 1$
- For any two events A and B,
  P(exactly one of A, B occur)
  $= P(A \cap \overline{B}) + P(B \cap \overline{A})$
  $= P(A) + P(B) - 2P(A \cap B)$
  $= P(A \cup B) - P(A \cap B)$
  $= P(A^c \cup B^c) - P(A^c \cap B^c)$
- **De Morgan's Law :** If A and B are any two subsets of a universal set U, then
  (a) $(A \cup B)^c = A^c \cap B^c$ and
  (b) $(A \cap B)^c = A^c \cup B^c$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For any three events A, B and C, we have



- $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$
  $- P(A \cap B) - P(B \cap C) - P(C \cap A)$
  $+ P(A \cap B \cap C)$
- P(at least two of A, B, C occur)
  $= P(B \cap C) + P(C \cap A) + P(A \cap B)$
  $- 2P(A \cap B \cap C)$
- P(exactly two of A, B, C occur)
  $= P(B \cap C) + P(C \cap A) + P(A \cap B)$
  $- 3P(A \cap B \cap C)$
- P(exactly one of A, B, C occurs)
  $= P(A) + P(B) + P(C) - 2P(B \cap C)$
  $- 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

REMARKS :

If three events A, B and C are pairwise mutually exclusive then they must be mutually exclusive.
i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0$
$\Rightarrow P(A \cap B \cap C) = 0$.
However the converse of this is not true.

Independent events : Two events A and B are said to be independent if occurrence or non occurrence of one does not affect the probability of the occurrence or non occurrence of other.

- If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be dependent.
  For two independent events A and B :
  $P(A \cap B) = P(A) \cdot P(B)$. Often this is taken as the definition of independent events.
- Three events A, B and C are independent if and only if all the following conditions hold;
  $P(A \cap B) = P(A) \cdot P(B)$;
  $P(B \cap C) = P(B) \cdot P(C)$
  $P(C \cap A) = P(C) \cdot P(A)$
  and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
  i.e. they must be pairwise as well as mutually independent .
- The probability of getting exactly r success in n independent trials is given by
  $P(r) = {}^nC_r \, p^r \, q^{n-r}$, where p = probability of success in a single trial and q = probability of failure in a single trial.
  Note : $p + q = 1$.
  Remark: Independent events are not in general mutually exclusive and vice versa.
  Mutually exclusiveness can be used when the events are taken from the same experiment and independence can be used when the events are taken from different experiments .

BAYE'S THEOREM OR TOTAL PROBABILITY THEOREM

If an event A can occur only with one of the n mutually exclusive and exhaustive events B_1, B_2, ..., B_n and the probabilities $P(A/B_1)$, $P(A/B_2)$, ..., $P(A/B_n)$ are known, then

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

Proof: The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, \ldots\ldots, B_n$

$\therefore \quad A = AB_1 + AB_2 + AB_3 + \ldots\ldots + AB_n$

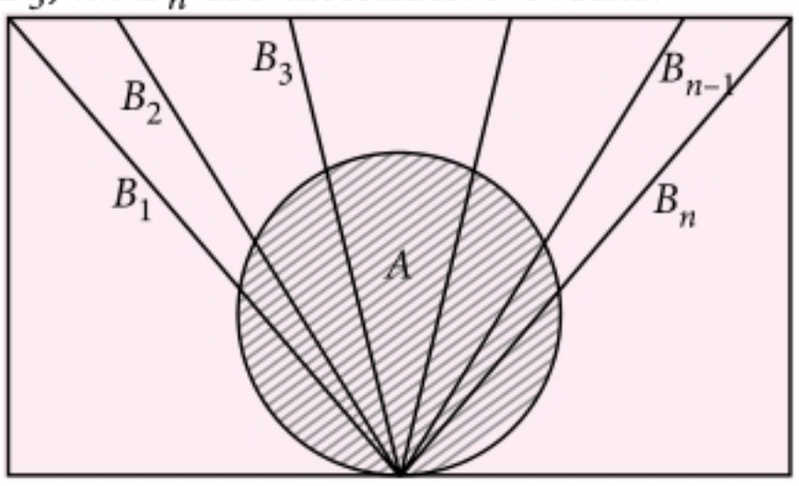
$$P(A) = P(AB_1) + P(AB_2) + \ldots\ldots + P(AB_n)$$

$$= \sum_{i=1}^{n} P(AB_i)$$

Remark: $A \equiv$ event what we have;

$B_1 \equiv$ event what we want ;

$B_2, B_3, \ldots. B_n$ are alternative events.



Now,

$$P(AB_i) = P(A)\cdot P(B_i/A) = P(B_i)\cdot P(A/B_i)$$

$$P(B_i / A) = \frac{P(B_i)\cdot P(A / B_i)}{P(A)} = \frac{P(B_i)\cdot P(A / B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

$$P(B_i / A) = \frac{P(B_i)\cdot P(A / B_i)}{\sum P(B_i)\cdot P(A / B_i)}$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. There are 12 railway stations in a railway line. A train stops at 4 stations randomly. Then probability that exactly three of them are consecutive is

(a) $\frac{28}{165}$ (b) $\frac{8}{55}$ (c) $\frac{8}{65}$ (d) $\frac{4}{55}$

2. If three one by one squares are selected at random from the chessboard, then probability that they form the letter 'L' is

(a) $\frac{49}{{}^{64}C_2}$ (b) $\frac{49}{{}^{63}C_2 + {}^{63}C_3}$

(c) $\frac{147}{63\cdot({}^{64}C_2)}$ (d) $\frac{98}{{}^{64}C_3}$

3. A and B are any two events of an experiment such that $0 < P(A), P(B) < 1$. If $P(B^c) > P(A^c)$, then

(a) $P(A \cap B^c) < P(A^c \cap B)$

(b) $P(A \cap B^c) = P(A^c \cap B)$

(c) $P(B \mid A) < P(A \mid B)$

(d) $P(B \mid A) > P(A \mid B)$

4. Two natural numbers a and b are selected at random. The probability that $(a^2 + b^2)$ is divisible by 7 is

(a) 3/8 (b) 1/7 (c) 3/49 (d) 1/49

5. A natural number x is chosen from the first 100 natural numbers. The probability that $\frac{x^2 - 60x + 800}{x - 30} < 0$ is

(a) 3/25 (b) 1/50

(c) 7/25 (d) none of these

6. Let A, B, C be three events. If the probability of occurring of exactly one event out of A and B is $1 - a$, out of B and C is $1 - 2a$, out of C and A is $1 - a$ and that of occurring, three events simultaneously is a^2, then the probability that at least one out of A, B, C will occur is

(a) 1/2 (b) greater than 1/2

(c) less than 1/2 (d) none of these

7. The probability that when 12 distinct balls are distributed among three boxes, the first will contain three balls is

(a) $\frac{2^9}{3^{12}}$ (b) $\frac{{}^{12}C_9 \cdot 2^9}{9^6}$

(c) $\frac{{}^{12}C_3 \cdot 2^{12}}{3^{12}}$ (d) none of these

8. There are 4 defective items in a lot consisting of 10 items. From this lot we select 5 items at random. The probability that there will be 2 defective items among them is

(a) 1/2 (b) 2/5 (c) 5/21 (d) 10/21

9. Box I contains 3 white, 1 black; Box II contains 2 white, 2 black and Box III contains 1 white, 3 black balls. If from each of these boxes one ball is drawn at random, find the probability that 2 white, 1 black balls will be drawn.

(a) 13/32 (b) 1/4 (c) 3/16 (d) 1/32

10. In a game of chance a player throws a pair of dice and scores points equal to the difference

between the numbers on the two dice. Winner is the person who scores exactly 5 points more than his opponent. If two players are playing this game only one time, then the probability that neither of them wins, is

(a) $\frac{1}{54}$ (b) $\frac{1}{108}$ (c) $\frac{53}{54}$ (d) $\frac{107}{108}$

11. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was

(a) $\frac{7}{2^{13}}$ (b) $\frac{1}{2^{13}}$ (c) $\frac{13}{14}$ (d) $\frac{13}{2^{13}}$

12. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2 respectively. If they race three times the probability that "the same horse wins all the three races" and the probability that a, b, c each wins one race, are respectively

(a) $\frac{8}{50}, \frac{9}{50}$ (b) $\frac{16}{100}, \frac{3}{100}$ (c) $\frac{12}{50}, \frac{15}{50}$ (d) $\frac{10}{50}, \frac{8}{50}$

13. Events A and C are independent. If the probabilities relating A, B and C are $P(A) = 1/5$; $P(B) = 1/6$; $P(A \cap C) = 1/20$; $P(B \cup C) = 3/8$, then

(a) events B and C are independent
(b) events B and C are mutually exclusive
(c) events B and C are neither independent nor mutually exclusive
(d) events B and C are equiprobable

14. Assume that the birth of a boy or a girl to a couple to be equally likely, mutually exclusive, exhaustive and independent of the other children in the family. For a couple having 6 children, the probability that their "three oldest are boys" is

(a) 20/64 (b) 1/64
(c) 8/64 (d) none of these

15. Box A contains 3 red and 2 blue marbles while box B contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up head, a marble is drawn from A, if it turns up tail, a marble is drawn from bag B. The probability that a red marble is chosen, is

(a) 1/5 (b) 2/5 (c) 3/5 (d) 1/2

16. On a Saturday night, 20% of all drivers in U.S.A. are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a saturday night smashed into a tree, the probability that the driver was under the influence of alcohol, is

(a) 3/7 (b) 4/7 (c) 5/7 (d) 6/7

17. Sixteen players s_1, s_2, ..., s_{16} play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. The probability that "exactly one of the two players s_1 and s_2 is among the eight winners" is

(a) 4/15 (b) 7/15 (c) 8/15 (d) 9/15

18. An even number of cards are drawn from a well shuffled pack of playing cards. The probability that half of them are red and the other half are black is

(a) $\frac{{}^{52}C_{26} - 1}{2^{51} - 1}$ (b) $\frac{{}^{52}C_{26} - 1}{2^{51}}$

(c) $\frac{{}^{52}C_{26}}{2^{51}}$ (d) $\frac{{}^{52}C_{26}}{2^{51} - 1}$

19. If A and B are two events such that, $P(A) = 0.3$, $P(B) = 0.25$, $P(A \cap B) = 0.2$, then $P\left(\left(\frac{A^c}{B^c}\right)^c\right)$ is equal to

(a) $\frac{2}{15}$ (b) $\frac{11}{15}$ (c) $\frac{13}{15}$ (d) $\frac{14}{15}$

20. Two real numbers x, y are chosen from the interval $[0, 8]$, then the probability that $y^2 > 2x$ is

(a) $\frac{1}{3}$ (b) $\frac{11}{32}$ (c) $\frac{11}{64}$ (d) $\frac{2}{3}$

21. Each of the 'n' urns contain 4 white and 6 black balls. The $(n+1)^{th}$ urn contains 5 white and 5 black balls. One of the urn is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n+1)^{th}$ urn was chosen to draw the balls is $\frac{1}{16}$ then the value of n is

(a) 10 (b) 11 (c) 12 (d) 13

22. Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let A be the event of selecting a point (x, y) satisfying $y^2 \le x$ and B be the event of selecting a point (x, y) satisfying $x^2 < y$, then

(a) $P(A \cap B) = \frac{1}{3}$ (b) $A \subset B$

(c) $2P(A) = 3P(B)$ (d) $P(B) < P(A)$

SECTION-II

More than One Correct Answer Type

23. For any two events A and B defined on a sample space, then

(a) $P(A/B) \ge \frac{P(A) + P(B) - 1}{P(B)}$, $P(B) \ne 0$ is always true.

(b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.

(c) $P(A \cup B) = 1 - P(A^c) \cdot P(B^c)$, if A and B are independent.

(d) $P(A \cup B) = 1 - P(A^c) \cdot P(B^c)$, if A and B are disjoint.

24. If E_1 and E_2 are two events such that $P(E_1) = 1/4$, $P(E_2/E_1) = 1/2$ and $P(E_1/E_2) = 1/4$, then

(a) E_1 and E_2 are independent.

(b) E_1 and E_2 are exhaustive.

(c) E_2 is twice as likely to occur as E_1.

(d) Probabilities of the events $E_1 \cap E_2$, E_1 and E_2 are in G.P.

25. From a pack of cards, a man is drawing cards one by one without replacement. Consider the events

E_1 : The event that the 10^{th} card drawn is second heart card.

E_2 : The event that the 8^{th} card drawn is third ace.

E_3 : The event that the 9^{th} card drawn is first king.

E_4 : The event that the 12^{th} card drawn is club '6'.

(a) $P(E_1) = \frac{{}^{39}C_8}{{}^{51}C_9} \times 3$ (b) $P(E_2) = \frac{594}{7735}$

(c) $P(E_3) = \frac{4({}^{48}C_8)}{9({}^{52}C_9)}$ (d) $P(E_4) = \frac{1}{52}$

26. If n different objects are distributed among $(n + 2)$ persons, then

(a) Probability that exactly 2 persons will get nothing is $\frac{(n+1)!}{2 \cdot (n+2)^{n-1}}$

(b) Probability that exactly 3 persons will not get anything is $\frac{{}^{n+2}C_3 (n-1)({}^{n}C_2)(n-2)!}{(n+2)^n}$

(c) Probability that exactly 3 persons will not get anything is $\frac{n \cdot (n-1)^2 \cdot (n+1)}{12(n+2)^{n-1}}$

(d) Probability that exactly 2 persons will get nothing is $\frac{n^2 . (n-1)^2}{(n+2)^{n-1}}$

27. If three numbers are chosen randomly from the set $\{1, 3, 3^2, ..., 3^n\}$ without replacement, then the probability that they form an increasing geometric progression is

(a) $\frac{3}{2n}$ if n is odd (b) $\frac{3}{2n}$ if n is even

(c) $\frac{3n}{2(n^2 - 1)}$ if n is even (d) $\frac{3n}{2(n^2 - 1)}$ if n is odd

28. A rational number is selected at random from the set of all rational numbers from the interval (2010, 2011) all whose digits after the decimal point are non -zero and are in the decreasing order, then

(a) The probability that it has exactly seven digits after the decimal point is $\frac{36}{511}$.

(b) The probability that it contains the digit 3 after the decimal point is $\frac{256}{511}$.

(c) The probability that the last digit after the decimal point is atleast 4 is $\frac{63}{511}$.

(d) The probability that the last digit after the decimal point is 4 is $\frac{32}{511}$.

29. Let a, b, c be integers and $a^2 + b^2 + c^2 = 2$.

Let $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. Consider the system of simultaneous equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(a) The probability that the system of equations has unique solution is 1/2.

(b) The number of triplets (a, b, c) for which the system of equations has infinitely many solutions is 6.

(c) If $a = 0$, the number of ordered pairs (b, c) for which the system of equations has no solution is 2.

(d) The number of elements in the range of $ab + bc + ca$ is 2.

30. If A and B are two independent events such that $P(A) = 1/2$ and $P(B) = 1/5$, then

(a) $P(A \cup B) = \frac{3}{5}$ (b) $P(A/B) = \frac{1}{2}$

(c) $P(A/A \cup B) = \frac{5}{6}$ (d) $P(A \cap B/A' \cup B') = 0$

31. A random variable x follows binomial distribution with mean 'a' and variance 'b'. Then

(a) $a > b > 0$ (b) $\frac{a}{b} > 1$

(c) $\frac{a^2}{a-b}$ is an integer (d) $\frac{a^2}{a+b}$ is an integer

32. Two fair dice are thrown, the probability that difference between the numbers is

(a) two is 2/9 (b) three is 1/6

(c) zero is 2/3 (d) none of these

33. For two events A and B, $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$. Then which of the following is/are correct?

(a) $P\left(\frac{A'}{B}\right) = \frac{3}{4}$ (b) $P\left(\frac{B'}{A'}\right) = \frac{1}{2}$

(c) $P(A \cup B) = \frac{3}{4}$ (d) $P(A \cap B) = \frac{1}{8}$

34. If p and q are chosen randomly from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with replacement, then the probability that the roots of the equation $x^2 + px + q = 0$ are

(a) real is $\frac{33}{50}$ (b) imaginary is $\frac{19}{50}$

(c) real and equal is $\frac{3}{100}$

(d) real and distinct is $\frac{59}{100}$

35. If A, B are exhaustive events in a sample space such that the probabilities of the events $A \cap B$, A, B and $A \cup B$ are in A.P. If $P(A) = K$, where $0 < K \leq 1$, then

(a) $P(B) = \frac{K+1}{2}$ (b) $P(A \cap B) = \frac{3K-1}{2}$

(c) $P(A \cup B) = 1$

(d) $P(A' \cup B') = \frac{3(1-K)}{2}$

36. If M and N are any two events, the probability that exactly one of them occurs is

(a) $P(M) + P(N) - 2P(M \cap N)$

(b) $P(M) + P(N) - P(M \cap N)$

(c) $P(M^c) + P(N^c) - 2P(M^c \cap N^c)$

(d) $P(M \cap N^c) + P(M^c \cap N)$

37. E and F are two independent events. The probability that both E and F happen is 1/12 and the probability that neither E nor F happens is 1/2. Then,

(a) $P(E) = 1/3, P(F) = 1/4$

(b) $P(E) = 1/2, P(F) = 1/6$

(c) $P(E) = 1/6, P(F) = 1/2$

(d) $P(E) = 1/4, P(F) = 1/3$

SECTION-III

Comprehension Type

Paragraph for Question No. 38 to 40

Let there be 'n' numbered objects $a_1, a_2, a_3, \ldots, a_n$. An arrangement of $a_1, a_2, \ldots, a_n$ taken all at a time is called a permutation. A permutation in which a_i does not at its place $(i = 1, 2, \ldots, n)$ is called a derangement. Let S_n be the number of those permutations of $a_1, a_2, \ldots, a_n$ which are deranged. Then $S_1 = 0$; $S_2 = 1$; $S_3 = 2$

38. The probability that number of only three objects out of $a_1, a_2, \ldots\ldots, a_7$ occupy their original position when they are arranged at random is

(a) $\frac{310}{7!}$ (b) $\frac{315}{7!}$

(c) $\frac{305}{7!}$ (d) none of these

39. Given 4 pairs of gloves. They are distributed to four persons such that each person gets a right handed and a left handed glove. The probability that no person gets a pair is

(a) $\frac{216}{576}$ (b) $\frac{196}{576}$

(c) $\frac{24}{576}$ (d) none of these

40. If P_n be the probability that none of the 'n' objects of $a_1, a_2, \ldots\ldots, a_n$ occupy their original position, then

(a) $nP_n = (n-1)(P_{n-1}) + P_{n-2}$

(b) $nP_n = n(P_{n-1}) + P_{n-2}$

(c) $nP_n = (n-1)(P_{n-1} + P_{n-2})$

(d) $P_n = \frac{1}{2}(P_{n-1} + P_{n-2})$

Paragraph for Question No. 41 to 43

A box contains nine slips bearing the numbers –4, –3, –2, –1, 0, 1, 2, 3, 4 (one number for one slip)

41. A slip is chosen at random, its number is noted and replaced in the box. If this experiment is repeated 5 times, then the probability that exactly 3 numbers appear in the noted list is

(a) $\frac{{}^9C_3 \times 90}{9^5}$ (b) $\frac{{}^9C_3 \times 243}{9^5}$

(c) $\frac{{}^9C_3 \times 150}{9^5}$ (d) $\frac{{}^9C_3 \times 240}{9^5}$

42. If a slip is chosen at random from the box and the experiment is repeated nine times (with replacement) to form a 3 × 3 matrix. (The chosen numbers are arranged row wise). The probability that the matrix so formed is skew symmetric is

(a) $\left(\frac{1}{9}\right)^9$ (b) $\left(\frac{1}{9}\right)^6$ (c) $\left(\frac{1}{9}\right)^3$ (d) $\left(\frac{1}{9}\right)^4$

43. A slip is chosen at random from the box, its number is noted and replaced in the box. If the experiment is repeated 5 times then the probability that sum of five noted numbers is zero is

(a) $\frac{3941}{9^5}$ (b) $\frac{3923}{9^5}$

(c) $\frac{3978}{9^5}$ (d) $\frac{3951}{9^5}$

Paragraph for Question No. 44 to 46

In an objective paper there are two sections of 10 questions each. For 'section I' each question has 5 options and only one option is correct and for each question in 'section II' has 4 options with multiple answers and marks for a question in this section is awarded only if he ticks all correct answers. Marks for each question in 'section I' is 1 and in 'section II' is 3 (There is no negative marking).

44. If a candidate attempts only two questions, one from 'section I' and one from 'section II'. The probability that he scores in both questions is

(a) $\frac{74}{75}$ (b) $\frac{1}{25}$ (c) $\frac{1}{15}$ (d) $\frac{1}{75}$

45. If a candidate attempts 4 questions in total, then the probability of scoring 10 marks is

(a) $\frac{1}{5}\left(\frac{1}{15}\right)^3$ (b) $\frac{4}{5}\left(\frac{1}{15}\right)^3$

(c) $\frac{1}{5}\left(\frac{14}{15}\right)^3$ (d) none of these

46. The probability of getting a score less than 40 in this paper is

(a) $\left(\frac{1}{75}\right)^{10}$ (b) $1-\left(\frac{1}{75}\right)^{10}$

(c) $\left(\frac{74}{75}\right)^{10}$ (d) $\left(\frac{1}{5}\right)^{10}$

Paragraph for Question No. 47 to 49

A coin shows head with probability p, $0 < p < 1$. X is a random variable which denotes the number of tosses before the appearance of the first head.
Thus TTTH $\Rightarrow X = 3$ and H $\Rightarrow X = 0$
If $P(X = n)$ denotes the probability of the first head appearing at the $(n+1)^{th}$ trial.

47. An expression for $P(X = 5)$ is

(a) $(1-p)^5 \cdot p$ (b) $(1-p)^4 \cdot p$

(c) $(1-p)^4$ (d) $(1-p)^5$

48. An expression for $P(3 \le X \le 6)$ is

(a) $(1-p)^3 - (1-p)^6$ (b) $(1-p)^2 - (1-p)^7$

(c) $(1-p)^2 - (1-p)^6$ (d) $(1-p)^3 - (1-p)^7$

49. An expression for $P(X \ge 4)$ is

(a) $p + (1-p)$ (b) $(1-p)^2$

(c) $1 - (1-p)^4$ (d) None of these

Paragraph for Question No. 50 to 52

A player tosses a coin and scores one point for every head and two points for every tail that turns up. He plays on until his score reaches or passes n. If P_n denotes the probability of getting a score of exactly n. Then

50. Value of P_n for $n \geq 1$ is

(a) $\frac{1}{3}\left\{2+\frac{(-1)^n}{2^n}\right\}$ (b) $\frac{1}{4}\left\{3+\frac{(-1)^n}{3^n}\right\}$

(c) $\frac{1}{5}\left\{4+\frac{(-1)^n}{2^n}\right\}$ (d) $\frac{1}{6}\left\{3+\frac{(-1)^n}{5^n}\right\}$

51. Value of P_3 is

(a) 1/2 (b) 5/8

(c) 1/3 (d) 1/6

52. Value of P_4 is

(a) 1/5 (b) 1/7

(c) 11/16 (d) 7/11

SECTION-IV

Matrix-Match Type

53. Match the probabilities in Column-I with their equivalents in Column-II.

Column I		Column II	
(A)	$P\left(\frac{A}{B}\right)+P\left(\frac{\bar{A}}{B}\right)=$	(p)	1
(B)	$P(A)\cdot P\left(\frac{B}{A}\right)+P(\bar{A})\cdot P\left(\frac{B}{\bar{A}}\right)=$	(q)	$2P(A)\cdot P(B)$
(C)	$P(A-B)+P(B-A)=$	(r)	$P(B)$
(D)	$P(A \cup B) - P(A \cap B) =$	(s)	$P(A\cap\bar{B})+P(\bar{A}\cap B)$
		(t)	0

54. A is a set containing n elements. A subset P of A is chosen at random. Then A is reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Match the following events with their probabilities:

Column I		Column II	
(A)	$P \cap Q = \phi$	(p)	$n\cdot 3^{n-1}/4^n$
(B)	$P \cap Q$ is a singleton set	(q)	$(3/4)^n$
(C)	$P \cup Q = A$	(r)	$C(2n, n)/4^n$
(D)	$n(P) = n(Q)$	(s)	$\frac{9n(n-1)}{2\cdot 4^n}$
		(t)	$(1/2)^n$

55. Match the following:

Column I		Column II	
(A)	Six different balls are put in 3 different boxes, no box being empty. The probability of putting balls in boxes in equal nos. is	(p)	20/27
(B)	Six letters are posted in 3 letter boxes. The probability that no letter box remains empty is	(q)	1/6
(C)	Two person A and B throw two dice each. If A throws a sum of 9, then the probability of B throwing a sum greater than A is	(r)	2/7
(D)	If A and B are independent events and $P(A) = 0.3$ and $P(A\cup\bar{B})=0.8$, then $P(B) =$	(s)	1/4

SECTION-V

Integer Answer Type

56. The probability of a bomb hitting a bridge is 1/2 and two direct hits are needed to destroy it. The least number of bombs required so that the probability of the bridge being destroyed is greater than 0.9 is

57. 21 tickets are numbered from 1 to 21 serially. Three are drawn at random. The chance that the numbers on the selected tickets are in A.P is 'P'. Then $\left[\frac{1}{P}-7\right]$ = ... where [.] is G.I.F.

58. The probability of man hitting a target in one fire is 1/5. Then the minimum number of fire he must follow in order to make his chance of hitting the target more than 3/4 is

59. Three natural numbers are taken at random from the set $A = \{x|1 \leq x \leq 100, x \in N\}$ and the probability that the A.M. of the numbers taken is 25

is $\frac{K}{{}^{100}C_3}$, then the digit in units place of K is

60. A box contains 9 balls. One ball is selected at random and is known to be white. If the probability that the box contains exactly K white balls is 1/9, then $K =$

61. A bag contains n identical red balls, $2n$ identical black balls and $3n$ identical white balls. If probability of drawing n balls of same colour is greater than or equal to 1/6, then minimum number of red balls in the bag is equal to _____

62. Let $P(A) = 0.4$ and $P(A \cup B) = P(A \cap B)$, then $5P(B) =$ _____

63. A letter is selected from each of the words "ARTICLE and TRIANGLE". The probability of selecting same letter is $k/28$, then $k =$ _____

64. A man throws a die until he gets a number bigger than 3. The probability that he gets 5 in the last throw is a/b, then $a + b =$ _____

65. A random variable X has poisson distribution with mean 2. Then $P(X > 1.5) = 1 - \frac{k}{e^2}$, then $k =$

SOLUTIONS

1. (b) : Here $n(S) = {}^{12}C_4$ and
$n(E) = 2(8) + 8(7) = 72$

$\therefore \quad P(E) = \frac{72}{{}^{12}C_4} = \frac{8}{55}$

2. (b) : In every pair of adjacent columns we have seven sets of three squares forming L shapes and there are seven such sets of columns 1st and 2nd; 2nd and 3rd ; ...; 7th and 8th
No. of favourable cases $= 7 \times 7 = 49$
Total number of ways three squares be chosen $= {}^{64}C_3$

$\therefore \quad$ Probability $= \frac{49}{{}^{64}C_3}$

Now we have, ${}^{64}C_3 = {}^{64}C_{61}$
${}^{64}C_3 = {}^{63}C_2 + {}^{63}C_3$

$${}^{64}C_3 = \frac{62}{3} \cdot {}^{64}C_2 \quad \therefore \quad \frac{49}{{}^{64}C_3} = \frac{147}{62 \cdot ({}^{64}C_2)}$$

3. (c) : $P(B^c) > P(A^c)$ (Given)
$\Rightarrow \quad P(B) < P(A)$
$\Rightarrow \quad P(A \cap B^c) + P(A \cap B) = P(A) > P(B)$
$= P(A^c \cap B) + P(A \cap B)$
$\Rightarrow \quad P(A \cap B^c) > P(A^c \cap B).$
$\therefore$ (a) is false and so is (b)
Since $P(B) < P(A)$

$$\therefore \quad P(B/A) = \frac{P(A \cap B)}{P(A)} < \frac{P(A \cap B)}{P(B)} = P(A/B)$$

So, (c) is true.

4. (d) : Let $a = 7m + r$ and $b = 7n + s$, where $0 \le r, s \le 6$
Total number of ways = 49
Number of favourable ways = 1 (when $r = 0, s = 0$)
$\therefore \quad$ Req. probability = 1/49

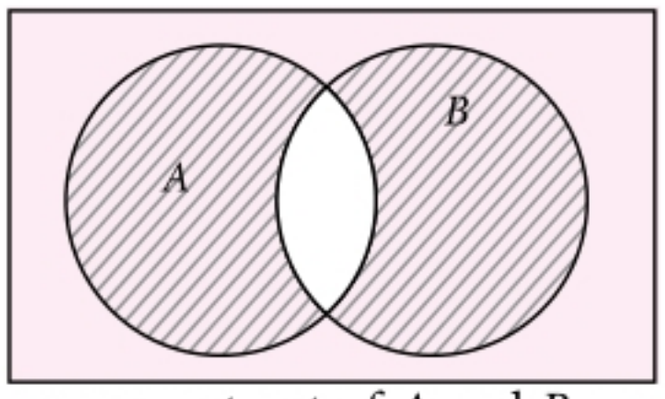
5. (c) : Since $\frac{x^2 - 60x + 800}{x - 30} < 0$

$$\Rightarrow \quad \frac{(x-20)(x-40)}{x-30} < 0$$

$\Rightarrow \quad x \in \{1, 2, ..., 19\} \cup \{31, 32, ..., 39\}$ [$\because x \in N$]
$\therefore \quad$ Number of favourable cases $= 19 + 9 = 28$
Total no. of cases = 100

$\therefore \quad$ Required probability $= \frac{28}{100} = \frac{7}{25}$

6. (b) :



P(exactly one event out of A and B occurs)
$= P[(A \cap B') \cup (A' \cap B)]$
$= P(A \cup B) - P(A \cap B)$
$= P(A) + P(B) - 2P(A \cap B)$
$\therefore \quad P(A) + P(B) - 2P(A \cap B) = 1 - a$ ...(1)
Similarly, $P(B) + P(C) - 2P(B \cap C) = 1 - 2a$ ...(2)
and $P(C) + P(A) - 2P(C \cap A) = 1 - a$ ...(3)
Also, $P(A \cap B \cap C) = a^2$ ...(4)
Now, P(atleast one out of A, B, C will occur) =
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2}[P(A) + P(B) - 2P(A \cap B) + P(B) + P(C)$
$- 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)]$
$+ P(A \cap B \cap C)$

$= \frac{1}{2}[1 - a + 1 - 2a + 1 - a] + a^2$

[using (1), (2), (3) and (4)]

$$= \frac{3}{2} - 2a + a^2 = \frac{1}{2} + (a-1)^2 > \frac{1}{2}$$

7. (b) : Number of ways when first box will contain three balls = ${}^{12}C_3 = {}^{12}C_9$

Favourable number of ways = ${}^{12}C_9 \times 2^9$

Total number of ways = 3^{12}

$\therefore$ Required probability = $\dfrac{{}^{12}C_9 \times 2^9}{9^6}$

8. (d) : [Hint: 10 $\langle$ 4D, 6G $\rightarrow$ 5 $\langle$ 2D, 3G

where D represent defective and G represent good items.

$\therefore$ Required probability = $\dfrac{{}^4C_2 \cdot {}^6C_3}{{}^{10}C_5} = \dfrac{10}{21}$

9. (a) :

Box I	Box II	Box III
3 W	2 W	1 W
1 B	2 B	3 B

2 white and 1 black ball can be chosen as

W W B + W B W + B W W

$\therefore$ Required Probability = $\left(\dfrac{3}{4} \times \dfrac{2}{4} \times \dfrac{3}{4}\right) + \left(\dfrac{3}{4} \times \dfrac{2}{4} \times \dfrac{1}{4}\right) + \left(\dfrac{1}{4} \times \dfrac{2}{4} \times \dfrac{1}{4}\right)$

$= \dfrac{18+6+2}{4 \times 16} = \dfrac{26}{4 \times 16} = \dfrac{13}{32}$

10. (c) : Player A can win if A throws (1, 6) or (6, 1) and B throws ((1, 1), (2, 2), (3, 3), (4, 4), (5, 5) or (6, 6)). Thus the number of ways is 12.

Similarly the number of ways in which B can win is 12.

$\therefore$ Total number of ways in which either A or B wins = 24

Thus the number of ways in which none of the two wins = $6^4 - 24$.

$\therefore$ Required probability = $\dfrac{6^4 - 24}{6^4} = \dfrac{53}{54}$

11. (a) : L and W can be filled at 14 places in 2^{14} ways, where L and W represent the losing and winning of the toss of a coin

$\therefore$ $n(S) = 2^{14}$

Now 13 L's and 1 W can be arranged at 14 places in 14 ways.

Hence $n(A) = 14$

$\therefore$ Required probability = $\dfrac{14}{2^{14}} = \dfrac{7}{2^{13}}$

12. (a) : $P(a) = 0.3$; $P(b) = 0.5$; $P(c) = 0.2$

$\Rightarrow$ a, b, c are exhaustive.

P(same horse wins all the three races)

$= P(aaa \text{ or } bbb \text{ or } ccc)$

$= (0.3)^3 + (0.5)^3 + (0.2)^3$

$= \dfrac{27+125+8}{1000} = \dfrac{160}{1000} = \dfrac{4}{25}$

P(each horse wins exactly one race)

$= P(abc \text{ or } acb \text{ or } bca \text{ or } bac \text{ or } cab \text{ or } cba)$

$= 0.3 \times 0.5 \times 0.2 \times 6 = 0.18 = \dfrac{9}{50}$

13. (a) : $P(A \cap C) = P(A) \cdot P(C)$

$\Rightarrow \dfrac{1}{20} = \dfrac{1}{5} P(C) \Rightarrow P(C) = \dfrac{1}{4}$

Now $P(B \cup C) = \dfrac{1}{6} + \dfrac{1}{4} - P(B \cap C)$

Hence, $P(B \cap C) = \dfrac{5}{12} - \dfrac{3}{8} = \dfrac{1}{24} = \dfrac{1}{6} \cdot \dfrac{1}{4} = P(B) \cdot P(C)$

14. (c) : $E : B_1\ B_2\ B_3 \times \times \times$ where $\times$ means B or G

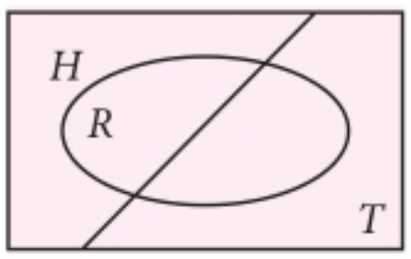
Hence $P(E) = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{8} = \dfrac{8}{64}$

15. (b) : $A \langle$ 3R, 2B ; $B \langle$ 2R, 8B

R = event that a red marble is drawn

$P(R) = P(R \cap H) + P(R \cap T)$

$= P(H)\ P(R/H) + P(T) \cdot P(R/T)$



$= \dfrac{1}{2}\left[\dfrac{3}{5} + \dfrac{2}{10}\right] = \dfrac{8}{10} \cdot \dfrac{1}{2} = \dfrac{2}{5}$

16. (c) : A : car met with an accident

B_1: driver was alcoholic, $P(B_1) = 1/5$

B_2: driver was sober, $P(B_2) = 4/5$

$P(A/B_1) = 0.001$; $P(A/B_2) = 0.0001$

Hence $P(B_1 / A) = \dfrac{(0.2)(0.001)}{(0.2)(0.001) + (0.8)(0.0001)} = \dfrac{5}{7}$

17. (c) : 7 players (leaving s_1 and s_2) out of 14 can be selected in ${}^{14}C_7$ ways and the 8^{th} player can be chosen in two ways *i.e.* either s_1 or s_2. Hence the total number of ways = ${}^{14}C_7 \cdot 2$

$$\therefore \quad \text{Required probability} = \frac{2 \cdot {}^{14}C_7}{{}^{16}C_8} = \frac{8}{15}$$

Alternative method : Let

E_1 : s_1 and s_2 are in the same group

E_2 : s_1 and s_2 are in the different group

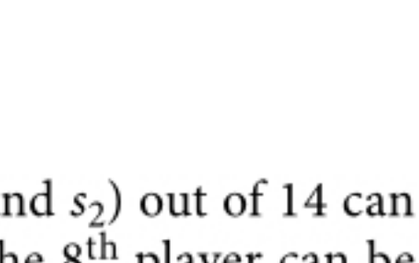
E : exactly one of the two players s_1 and s_2 is among the eight winners.

$$E = (E \cap E_1) \cup (E \cap E_2)$$

$$\Rightarrow \quad P(E) = P(E \cap E_1) + P(E \cap E_2)$$

$$\Rightarrow \quad P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$$

Now $P(E_1) = \dfrac{\frac{(14)!}{(2)^7 \cdot 7!}}{\frac{16!}{2^8 \cdot 8!}} = \dfrac{1}{15}$



$$P(E_2) = 1 - \frac{1}{15} = \frac{14}{15}$$

Hence $P(E) = \dfrac{1}{15} \cdot 1 + \dfrac{14}{15} \cdot P$ (exactly one of either S_1 or S_2 wins)

$$= \frac{1}{15} + \frac{14}{15} \cdot \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \right)$$

$$= \frac{1}{15} + \frac{14}{15} \cdot \frac{1}{2} = \frac{1}{15} + \frac{7}{15} = \frac{8}{15}$$

18. (a) : $n(E) = {}^{52}C_2 + {}^{52}C_4 + {}^{52}C_6 + \ldots + {}^{52}C_{52}$

$$= \frac{1}{2}(2^{52}) - 1 = 2^{51} - 1$$

$$m(E) = {}^{26}C_1 \cdot {}^{26}C_1 + {}^{26}C_2 \cdot {}^{26}C_2 + \ldots + {}^{26}C_{26} \cdot {}^{26}C_{26}$$

$$= {}^{52}C_{26} - 1$$

$$\therefore \quad P(E) = \frac{m}{n} = \frac{{}^{52}C_{26} - 1}{2^{51} - 1}$$

19. (a) : $P\left(\left(\dfrac{A^c}{B^c}\right)^c\right) = 1 - P\left(\dfrac{A^c}{B^c}\right)$

$$= 1 - \frac{P(A^c \cap B^c)}{P(B^c)} = 1 - \left\{ \frac{1 - P(A \cup B)}{1 - P(B)} \right\}$$

$$= 1 - \left\{ \frac{1 - (P(A) + P(B) - P(A \cap B))}{1 - P(B)} \right\}$$

$$= 1 - \frac{0.65}{0.75} = 1 - \frac{13}{15} = \frac{2}{15}$$

20. (d) : $n(S) = 8 \times 8 = 64 \quad (\because \ x, y \in [0, 8])$

$$n(E) = \int_0^8 (8 - \sqrt{2}\sqrt{x})\,dx = \left(8x - \frac{\sqrt{2}x^{3/2}}{3/2} \right)_0^8$$

$$= 64 - \frac{2\sqrt{2}}{3} 8 \times 2\sqrt{2} = 64\left(1 - \frac{1}{3}\right) = 64 \times \frac{2}{3}$$

$\therefore$ Required probability = 2/3

21. (a) : Let E_1 be the event that one of first n urns is chosen, E_2 be the event that $(n + 1)^{th}$ urn is chosen.

A be the event that two balls drawn are black.

$$\therefore \quad P(E_1) = \frac{n}{n+1}; \ P(E_2) = \frac{1}{n+1}$$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^6C_2}{{}^{10}C_2} = \frac{1}{3}; \ P\left(\frac{A}{E_2}\right) = \frac{{}^5C_2}{{}^{10}C_2} = \frac{2}{9}$$

Also, $P\left(\dfrac{E_2}{A}\right) = \dfrac{P(E_2) \cdot P\left(\dfrac{A}{E_2}\right)}{P(E_1) P\left(\dfrac{A}{E_1}\right) + P(E_2) \cdot P\left(\dfrac{A}{E_2}\right)}$

$$\Rightarrow \quad \frac{1}{16} = \frac{\frac{1}{n+1} \cdot \frac{2}{9}}{\frac{n}{n+1} \cdot \frac{1}{3} + \frac{1}{n+1} \cdot \frac{2}{9}} \Rightarrow n = 10$$

22. (a) : $P(A) = \dfrac{\int_0^1 \sqrt{x}\,dx}{\int_0^1 dx} = \frac{2}{3}; P(B) = \dfrac{\int_0^1 \sqrt{y}\,dy}{\int_0^1 dy} = \frac{2}{3}$

$$\therefore \quad P(A \cap B) = \frac{\int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx}{\int_0^1 dx} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

23. (a, b, c): Since $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

Let $P(A \cap B) \geq P(A) + P(B) - 1$

or $1 \geq P(A) + P(B) - P(A \cap B)$

or $1 \geq P(A \cup B)$

which is true $\Rightarrow$ (a) is correct, (b) and (c) are obvious.

24. (a, c, d) : $P(E_2 / E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)}$

$$\Rightarrow \quad \frac{1}{2} = \frac{P(E_1 \cap E_2)}{1/4} \quad \Rightarrow \quad P(E_1 \cap E_2) = \frac{1}{8}$$

$$= P(E_2)\cdot P(E_1/E_2) = P(E_2)\cdot\frac{1}{4}$$

$$\Rightarrow \quad P(E_2)=\frac{1}{2}$$

Since $P(E_1\cap E_2)=\frac{1}{8}=P(E_1)\cdot P(E_2)$

$\Rightarrow$ events are independent.

Also $P(E_1\cup E_2)=\frac{1}{2}+\frac{1}{4}-\frac{1}{8}=\frac{5}{8}$

$\Rightarrow$ E_1 and E_2 are non exhaustive.

25. (a, c, d) : $P(E_1)= {}^9C_1\times\frac{13}{52}\times\frac{39}{51}\times\frac{38}{50}\times\frac{37}{49}\times\frac{36}{48}$

$$\times\frac{35}{47}\times\frac{34}{46}\times\frac{33}{45}\times\frac{32}{44}\times\frac{12}{43}=3\times\frac{{}^{39}C_8}{{}^{51}C_9}$$

$$P(E_2)={}^7C_2\times\frac{4}{52}\times\frac{3}{51}\times\frac{48}{50}\times\frac{47}{49}\times\frac{46}{48}\times\frac{45}{47}$$

$$\times\frac{44}{46}\times\frac{2}{45}=\frac{132}{38675}$$

$$P(E_3)=\frac{48\times47\times.....\times41}{52\times51\times.....\times45}\times\frac{4}{44}=\frac{4}{9}\left(\frac{{}^{48}C_8}{{}^{52}C_9}\right)$$

$$P(E_4)=\frac{51\times50\times.....\times41}{52\times51\times.....\times42}\times\frac{1}{41}=\frac{1}{52}$$

26. (a, b) : Probability that exactly 2 persons will get nothing

$$= {}^{n+2}C_2\,\frac{n!}{(n+2)^n}=\frac{(n+2)(n+1)n!}{2\cdot(n+2)^n}=\frac{(n+1)!}{2(n+2)^{n-1}}$$

Probability that exactly 3 persons will get nothing

$$= {}^{n+2}C_3\cdot\frac{{}^{n-1}C_1\cdot{}^nC_2\cdot(n-2)!}{(n+2)^n}$$

27. (a, c) : Number of triplets

$(3^r, 3^{r+1}, 3^{r+2})$ $(0\le r\le n-2)$ is $n-1$

Number of triplets

$(3^r, 3^{r+2}, 3^{r+4})(0\le r\le n-4)$ is $n-3$

. . .

. . .

. . .

. . .

Number of triplets,

$\left(3^r, 3^{r+\frac{n-1}{2}}, 3^{r+n-1}\right)(n \text{ odd})$ is 2

Number of triplets,

$\left(3^r, 3^{r+\frac{n}{2}}, 3^{r+n}\right)(n \text{ even})$ is 1

$\therefore$ If n is odd, the number of favourable outcomes

$$=(n-1)+(n-3)+....+4+2=\frac{n^2-1}{4}$$

and if n is even, the number of favourable outcomes

$$=(n-1)+(n-3)+....+3+1=\frac{n}{2}\times\frac{n}{2}=\frac{n^2}{4}$$

$\therefore$ Required Probability

$$=\frac{(n^2-1)/4}{{}^{(n+1)}C_3}=\frac{3}{2n}, \text{ if } n \text{ is odd}$$

or $\quad\frac{n^2/4}{{}^{(n+1)}C_3}=\frac{3n}{2(n^2-1)}$ if n is even

28. (a, b, c, d) : $n(S) = {}^9C_1 + {}^9C_2 + ... + {}^9C_9$

$= 2^9 - 1 = 511$

1) $n(E) = {}^9C_7 \times 1 = 36$
2) $n(E) = {}^8C_0 + {}^8C_1 + ... + {}^8C_8 = 2^8 = 256$
3) $n(E) = {}^6C_1 + {}^6C_2 + ... + {}^6C_6 = 2^6 - 1 = 63$
4) $n(E) = {}^5C_0 + {}^5C_1 + ... + {}^5C_5 = 2^5 = 32$

29. (a, c, d) : $a^2 + b^2 + c^2 = 2$ and a, b, c are integers

$\Rightarrow$ $a = 0, b = \pm1, c = \pm1$ or $a = \pm1, b = 0, c = \pm1$ or $a = \pm1, b = \pm1, c = 0$.

There are 12 possibilities.

1) Now $|A|\neq0 \Rightarrow a+b+c\neq0 \Rightarrow$ 6 possibilities $\Rightarrow$ probability = 1/2
2) $|A| = 0 \Rightarrow a+b+c=0 \Rightarrow 6$ possibilities

But in no case, the system has infinitely many solution.

3) If $a = 0, b = 1, c = -1$; $a = 0, b = -1, c = 1$.

$\Rightarrow$ Two possibilities

4) Range = $\{-1, 1\}$ only

30. (a, b, c, d) : $P(A\cup B) = 1 - P(A')\,P(B')$

$$= 1-\frac{1}{2}\cdot\frac{4}{5}=\frac{3}{5}$$

$$P(A/B)=P(A)=\frac{1}{2}$$

$$P(A/A\cup B)=\frac{P(A)}{P(A\cup B)}=\frac{1/2}{3/5}=\frac{5}{6}$$

Since $A'\cup B'=(A\cap B)' \Rightarrow P\left(\frac{A\cap B}{A'\cup B'}\right)=0$

31. (a, b, c) : Given, $a = np$, $b = npq$

Since $0 < q < 1 \Rightarrow 0 < npq < np$

$\Rightarrow 0 < b < a \Rightarrow \frac{a}{b} > 1$

Also $\frac{a^2}{a-b} = \frac{n^2p^2}{np-npq} = \frac{np}{1-q} = n$, which is an integer.

32. (a, b) : Difference is two if the outcomes are: (1, 3), (2, 4), (3, 5), (4, 6), (3, 1), (4, 2), (5, 3), (6, 4), similarly for other.

33. (a, b, d) : $P(A) = P(A/B) = 1/4$

$\Rightarrow$ A and B are independent,

hence, $P(B/A) = P(B) = 1/2$

Now A', B'; A', B will also be independent

$$\therefore \quad P\left(\frac{A'}{B}\right) = P(A') = 1 - P(A) = \frac{3}{4}$$

$$P\left(\frac{B'}{A'}\right) = P(B') = 1 - P(B) = \frac{1}{2}$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 5/8$

$P(A \cap B) = 1/8$

34. (b, c, d) : $D = b^2 - 4ac = p^2 - 4q$ $\therefore$ roots are real and equal when $p^2 = 4q$

$\therefore$ (2, 1), (4, 4), (6, 9)

$\therefore$ Required probability $= \frac{3}{100}$

Roots are imaginary : $p^2 < 4q$

A	q	p	
$p^2 < 4$	1	1	1
$p^2 < 8$	2	1,2	3
$p^2 < 12$	3	1,2,3	6
$p^2 < 16$	4	1,2,3	9
$p^2 < 20$	5	1,2,3,4	13
$p^2 < 24$	6	1,2,3,4	17
$p^2 < 28$	7	1,2,3,4,5	22
$p^2 < 32$	8	1,2,3,4,5	27
$p^2 < 36$	9	1,2,3,4,5	32
$p^2 < 40$	10	1,2,3,4,5,6	38

$\therefore$ Probability $= \frac{38}{100} = \frac{19}{50}$

P (roots are real) $= \frac{62}{100} = \frac{31}{50}$

P (roots are real and distinct) $= \frac{59}{100}$

35. (a, b, c, d) : A, B are exhaustive events

$\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$

$2P(B) = P(A) + P(A \cup B) = K + 1$

$\Rightarrow P(B) = \frac{K+1}{2}$

$-P(A \cap B) = P(A \cup B) - P(A) - P(B)$

$= 1 - K - \frac{(K+1)}{2} = \frac{2 - 2K - K - 1}{2}$

$\Rightarrow P(A \cap B) = \frac{3K-1}{2}$

36. (a, c, d) : Given that M and N are any two events.

To check the probability that exactly one of them occurs. We check all the options one by one.

(a) $P(M) + P(N) - 2P(M \cap N)$

$= [P(M) + P(N) - P(M \cap N)] - P(M \cap N)$

$= P(M \cup N) - P(M \cap N)$

= Probability that exactly one of M and N occurs.

(b) $P(M) + P(N) - P(M \cap N) = P(M \cup N)$

= probability that at least one of M and N occurs.

(c) $P(M^c) + P(N^c) - 2P(M^c \cap N^c)$

$= 1 - P(M) + 1 - P(N) - 2P(M \cup N)^c$

$= 2 - P(M) - P(N) - 2[1 - P(M \cup N)]$

$= 2 - P(M) - P(N) - 2 + 2P(M \cup N)$

$= P(M \cup N) + P(M \cup N) - P(M) - P(N)$

$= P(M \cup N) - P(M \cap N)$

= Probability that exactly one of M and N occurs.

(d) $P(M \cap N^c) + P(M^c \cap N)$

= Probability that M occurs but not N or probability that M does not occur but N occurs.

= Probability that exactly one of M and N occurs.

37. (a, d) : Let $P(E) = x$ and $P(F) = y$

Acc. to ques., $P(E \cap F) = \frac{1}{12}$

As E and F are independent events

$\therefore \quad P(E \cap F) = P(E)P(F)$

$$\Rightarrow \quad \frac{1}{12} = xy \quad \Rightarrow \quad xy = \frac{1}{12} \qquad ...(1)$$

Also $P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F}) = 1 - P(E \cup F)$

$$\Rightarrow \quad \frac{1}{2} = 1 - [P(E) + P(F) - P(E)P(F)]$$

$$\Rightarrow \quad x + y - xy = \frac{1}{2} \Rightarrow \quad x + y = \frac{7}{12} \qquad ...(2)$$

Solving (1) and (2) we get

either $x = \frac{1}{3}, \ y = \frac{1}{4}$ or $x = \frac{1}{4}, \ y = \frac{1}{3}$

38. (b) : Three objects out of 7 can be selected in ${}^7C_3 = 35$ ways. Remaining four will not be in their positions in 9 ways.

$\therefore \quad n(E) = 35 \times 9 = 315, \ n(S) = 7!.$

Required probability $= \dfrac{315}{7!}$

39. (a) : First distribute right handed glove to four person in 4! ways. Then distributed left handed glove in such a way that no person get a pair in 9 ways.

$\therefore \quad n(E) = 4! \times 9 = 216 \ ; \ n(S) = (4!)^2 = 576$

$$\Rightarrow \quad P(E) = \frac{216}{576}$$

40. (a) : By using exclusion and inclusion principle that $P_n = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + \frac{(-1)^n}{n!}\right)$ it can be shown that option.

41. (c) : Required probability $=$

$$\frac{{}^9C_3 \cdot \left({}^3C_1 \cdot {}^2C_2 \cdot \frac{5!}{3!} + {}^3C_2 \cdot {}^1C_1 \cdot \frac{5!}{2!.2!}\right)}{9^5} = \frac{{}^9C_3 \cdot (60 + 90)}{9^5}$$

42. (b)

43. (d) : The coefficient of x^0 in $(x^{-4} + x^{-3} + x^{-2} + ... + x^4)^5$

The coefficient of x^0 in $\left(\dfrac{1 + x + x^2 + ... + x^8}{x^4}\right)^5$

Coefficient of x^{20} in $(1 - x^9)^5 \ . \ (1 - x)^{-5}$

$= {}^{24}C_{20} - 5^{15}C_{11} + 10.{}^4C_2 = 3951$

Required probability $= \dfrac{3951}{9^5}$

44. (d) : Required probability $= \frac{1}{5} \times \frac{1}{15} = \frac{1}{75}$

45. (a) : It can be possible when candidate select one question from section - I and three questions from section - II

$\therefore \quad$ Required probability $= \frac{1}{5}\left(\frac{1}{15}\right)^3$

46. (b) : Required probability $= 1 - \left(\frac{1}{5} \times \frac{1}{15}\right)^{10}$

47. (a) : The event associated with $X = 5$ is

T T T T T H

$\Rightarrow \quad P(X = 5) = (1 - p)^5 . p$

48. (d) : $P(3 \le X \le 6) = (1 - p)^3 \, p + (1 - p)^4 p + (1 - p)^5 p + (1 - p)^6 p$

$= (1 - p)^3 - (1 - p)^7$

49. (d)

50. (a) **51. (b)** **52. (c)**

The score of n can be achieved in the following mutually exclusive ways

(i) by throwing a head when the score is $(n - 1)$

(ii) by throwing a tail when the score is $(n - 2)$

$A_i =$ Getting score of exactly i, $i = 1, 2, ...$

$H =$ Getting head in a toss, $T =$ Getting tail in a toss

$$P(A_n) = P\{(A_{n-1} \cap H) \cup (A_{n-2} \cap T)\}$$
$$= P(A_{n-1})P(H) + P(A_{n-2})P(T)$$

$$\Rightarrow \quad P(A_n) = \frac{1}{2}(P_{n-1} + P_{n-2})$$

$$\Rightarrow \quad P_n + \frac{1}{2}P_{n-1} = P_{n-1} + \frac{1}{2}P_{n-2}$$
$$= P_{n-2} + \frac{1}{2}P_{n-3} = P_2 + \frac{1}{2}P_1$$

Since $P_1 = P(H) = \frac{1}{2}, P_2 = P(T \cup HH) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$$\Rightarrow \quad P_n + \frac{1}{2}P_{n-1} = 1$$

$$P_n + \frac{1}{2}P_{n-1} = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)$$

$$\Rightarrow \quad P_n - \frac{2}{3} = -\frac{1}{2}\left(P_{n-1} - \frac{2}{3}\right) = -\frac{1}{2}\left\{-\frac{1}{2}\left(P_{n-2} - \frac{2}{3}\right)\right\}$$

$$= \left(-\frac{1}{2}\right)^2\left(P_{n-2} - \frac{2}{3}\right) = \frac{1}{3}\left(-\frac{1}{2}\right)^n$$

$$\Rightarrow \quad P_n = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n = \frac{1}{3}\left\{2 + \frac{(-1)^n}{2^n}\right\} \text{ for } n \geq 1$$

53. A → p; B → r; C → s; D → s

(A) $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = \frac{P(A \cap B)}{P(B)} + \frac{P(\bar{A} \cap B)}{P(B)} = 1$

(B) $P(A) \cdot P\left(\frac{B}{A}\right) + P(\bar{A}) \cdot P\left(\frac{B}{\bar{A}}\right)$

$$= P(A) \cdot \frac{P(B \cap A)}{P(A)} + P(\bar{A}) \cdot \frac{P(B \cap \bar{A})}{P(\bar{A})}$$

$$= P(B \cap A) + P(B \cap \bar{A}) = P(B)$$

(C) $P(A - B) + P(B - A) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(D) $P(A \cup B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B) - P(A \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A \cap \bar{B}) + P(B \cap \bar{A})$

54. A → q; B → p; C → q ; D → r

(A) $n(E) = \sum_{r=0}^{n} {}^nC_r \cdot 2^{n-r} = 3^n$

(B) $n(E) = \sum_{r=1}^{n} {}^nC_r \, {}^rC_1 \cdot 2^{n-r} = n \cdot 3^{n-1}$

(C) $n(E) = \sum_{r=0}^{n} {}^nC_r \cdot (1)(2^r) = 3^n$

(D) $n(E) = \sum_{r=0}^{n} {}^nC_r \, {}^nC_r = {}^{2n}C_n$

55. A → q; B → p; C → q; D → r

(A) Total no. of ways of putting the balls in boxes

$$= \frac{6!}{2!2!2!} + \frac{6!3!}{1!2!4!} + \frac{6!3!}{1!2!3!} = 540$$

No. of favourable ways $= \frac{6!}{2!2!2!} = 90$

Required probability $= \frac{1}{6}$

(B) Probability $= \frac{3^6 - {}^3C_1 \cdot 2^6 + 3}{3^6} = \frac{20}{27}$

(C) No. of ways A can throw greater than 9 is 6

Required probability $= \frac{1}{6}$

(D) $1 - P(B) + P(A \cap B) = 0.8$

$$\Rightarrow \quad P(B)(.7) = .2 \Rightarrow P(B) = \frac{2}{7}$$

56. (7) : Let 'n' be the least number of bombs required and 'X' be the number of bombs that hit the bridge. Then 'X' follows a binomial distribution with parameters n and $p = 1/2$. Now

$P(X \geq 2) > 0.9$

$\Rightarrow \quad 1 - P(X < 2) > 0.9$

$\Rightarrow \quad P(X = 0) + P(X = 1) < 0.1$

$$\Rightarrow \quad {}^nC_0\left(\frac{1}{2}\right)^n + {}^nC_1\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right) < \frac{1}{10}$$

$$\Rightarrow \quad \frac{n+1}{2^n} < \frac{1}{10}$$

By trial and error, we get $n \geq 7$. Thus the least value of 'n' is 7.

57. (6) : For $2n + 1$ tickets :

$$P = \frac{1 + 3 + 5 + ... + (2n-1)}{{}^{2n+1}C_3}$$

$$P = \frac{3n}{4n^2 - 1} = \frac{10}{133} \text{ where } n = 10$$

$$\left[\frac{1}{P} - 7\right] = 6$$

58. (7) : Probability of hitting the target in one fire

$$p = \frac{1}{5}$$

Then probability of hitting the target at least once in n fires

= 1 – (probability of not hitting the target)

$$= 1 - \left(\frac{4}{5}\right)^n > \frac{3}{4}$$

$$\Rightarrow \quad \left(\frac{4}{5}\right)^n < \frac{1}{4} \text{ as } \left(\frac{4}{5}\right)^6 > \frac{1}{4} \text{ and } \left(\frac{4}{5}\right)^7 < \frac{1}{4}$$

The least value of $n = 7$

59. (1) : $n(S) = {}^{100}C_3$

Since A.M. of 3 numbers = 25

Then their sum is 75

$x_1 + x_2 + x_3 = 75, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

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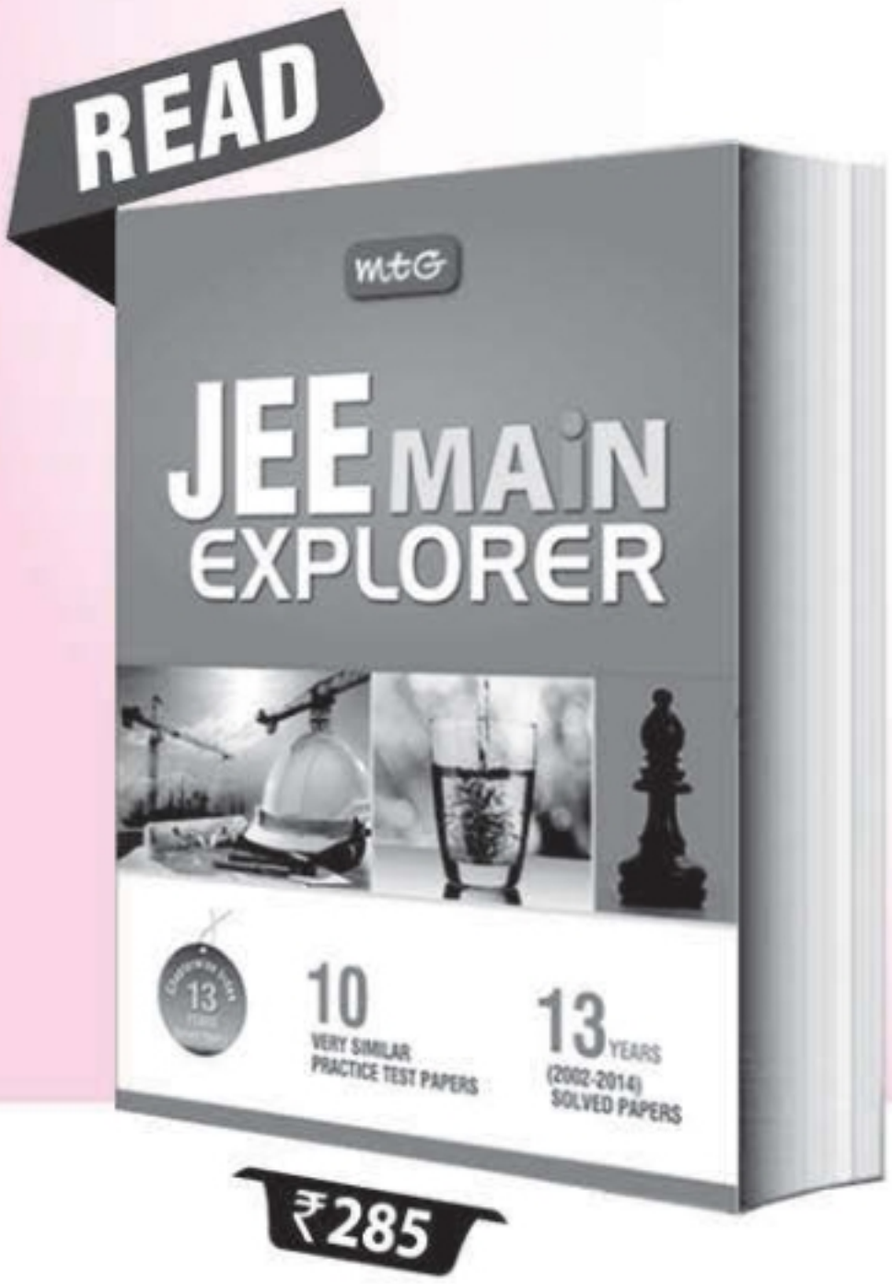

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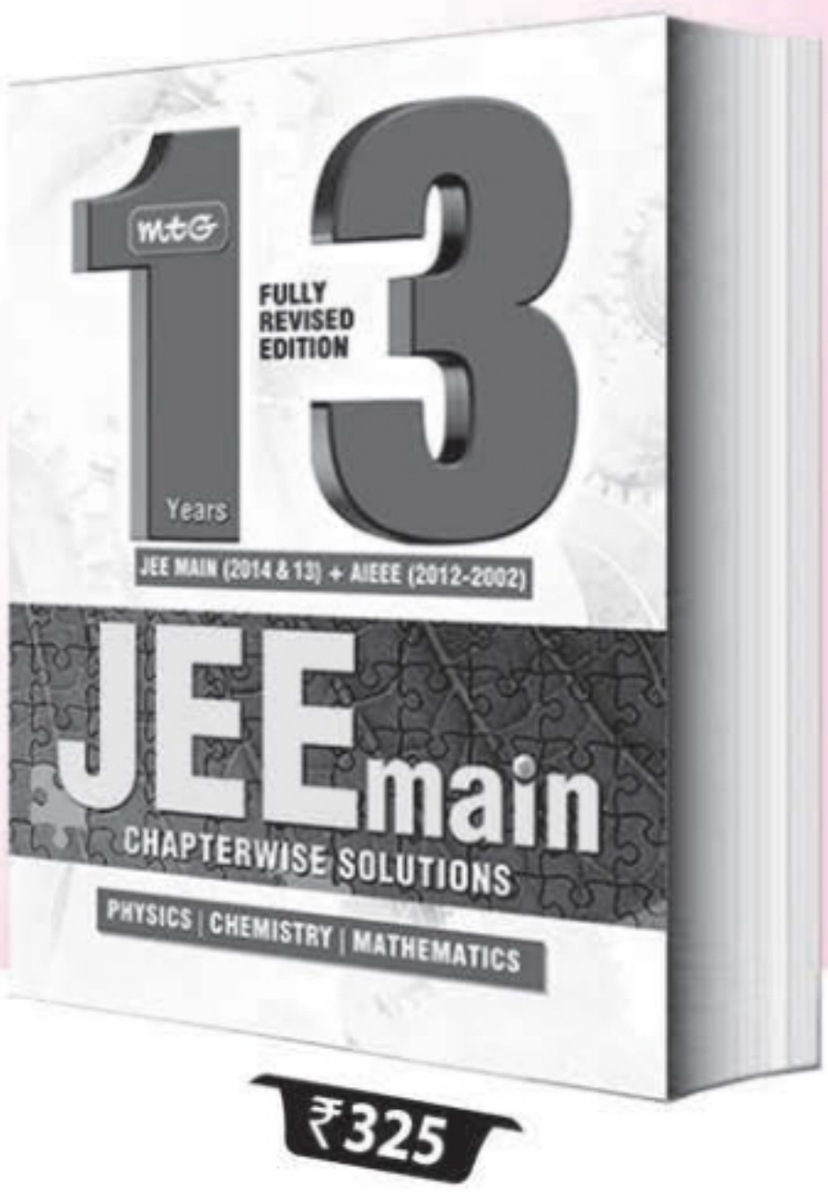

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No. of +ve integral solutions $= {}^{74}C_2 = 2701$

$\therefore$ Required probability $= \dfrac{2701}{{}^{100}C_3}$

60. (5) : $\dfrac{\frac{1}{9}\times\frac{K}{9}}{\frac{1}{9}\times\frac{1}{9}+....+\frac{1}{9}\times\frac{9}{9}} = \frac{1}{9}$

$$\Rightarrow \frac{K}{\left(\frac{9\times10}{2}\right)} = \frac{1}{9} \Rightarrow \frac{K}{45} = \frac{1}{9} \Rightarrow K = 5$$

61. (5) : Total number of cases = coefficient of x^n in $(1 + x + x^2 + ... + x^n)^3 = {}^{n+2}C_2$

So required probability $= \dfrac{3}{{}^{n+2}C_2}$

$$= \frac{8}{(n+1)(n+2)} \geq \frac{1}{6}$$

$$\Rightarrow n \geq \frac{-3+\sqrt{145}}{2}$$

$n_{\min} = 5$

So minimum number of red balls = 5.

62. (2) : Here $A = B \Rightarrow 5P(B) = 5 \times 0.4 = 2$

63. (3) : Common letters in the given words are A, R, T, I, L, E

Probability of selecting each of these letters $= \frac{1}{7}.\frac{1}{8}$

$\therefore$ Required probability $= 6\left(\frac{1}{7}.\frac{1}{8}\right) = \frac{3}{28}$

64. (4) : In the last throw : Out of 4, 5, 6 the favourable outcome is 5, Probability of getting 5 is $\frac{1}{3}$

$\therefore \quad a + b = 4$

65. (3) : $P(X > 1.5) = 1 - \{P(X = 0) + P(X = 1)\}$

$= 1 - \{e^{-2} + 2e^{-2}\}$

$= 1 - \dfrac{3}{e^2}$

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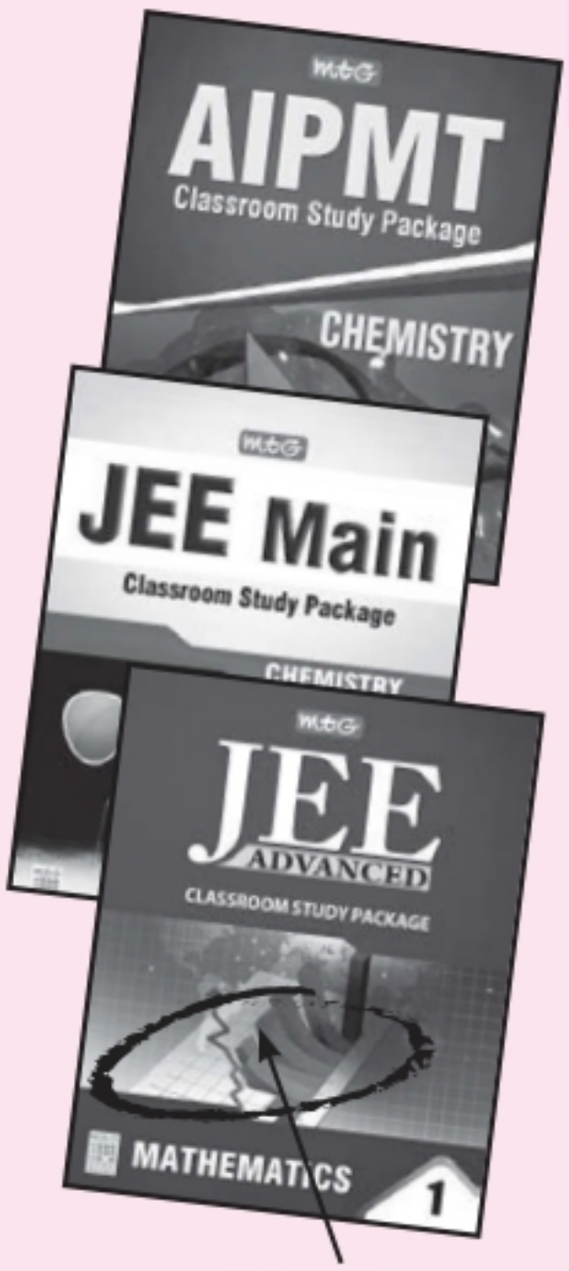

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Series- 6 **Integrals, Application of Integrals & Differential Equations**

INDEFINITE INTEGRALS

Definition : If $\frac{d}{dx}(\phi(x)) = f(x)$, then

$\int f(x)\,dx = \phi(x) + C$ ....(*) where C is any real number and is called constant of integration or arbitrary constant.

PROPERTIES OF INDEFINITE INTEGRALS

(a) The process of differentiation and integration are inverse of each other. *i.e.*, $\frac{d}{dx}\int f(x)dx = f(x)$

and $\int f'(x)dx = f(x)+C$, where C is any arbitrary constant.

(b) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent. So if f and g are two functions such that $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$, then

$\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

(c) The integral of the sum of two functions equals the sum of the integrals of the functions.

i.e., $\int (f(x)+g(x))dx = \int f(x)\,dx + \int g(x)dx$

(d) A constant factor may be written either before or after the integral sign.

i.e., $\int a\ f(x)\,dx = a\int f(x)\,dx$, where '$a$' is any constant.

(e) Properties (c) and (d) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int (k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x))\,dx =$$

$$k_1\int f_1(x)\,dx + k_2\int f_2(x)dx + ... + k_n\int f_n(x)\,dx$$

SOME STANDARD INTEGRALS

(i) $\int dx = x + C$

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$

(iii) $\int e^x\,dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log_e a} + C$, where $a > 0$

(v) $\int \frac{1}{x}dx = \log_e|x| + C$, where $x \neq 0$

(vi) $\int \sin x\,dx = -\cos x + C$

(vii) $\int \cos x\,dx = \sin x + C$

(viii) $\int \sec^2 x\,dx = \tan x + C$

(ix) $\int \text{cosec}^2 x\,dx = -\cot x + C$

(x) $\int \sec x \tan x\,dx = \sec x + C$

(xi) $\int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + C$

(xii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$,

where $|x| < 1$

(xiii) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$

(xiv) $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C$ or $-\operatorname{cosec}^{-1}x + C$

(xv) $\int \tan x\,dx = \log|\sec x| + C$ or $-\log|\cos x| + C$

(xvi) $\int \cot x\,dx = \log|\sin x| + C$

(xvii) $\int \sec x\,dx = \log|\sec x + \tan x| + C$ or $\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

(xviii) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$ or $\log\left|\tan\frac{x}{2}\right| + C$

METHODS OF INTEGRATION

(a) Integration by Substitution

(b) Integration using Partial Fractions

(c) Integration by Parts

(a) Integration by Substitution

	Integrals	Substitutions
•	$\int (ax+b)\sqrt{cx+d}\,dx$ or $\int \frac{ax+b}{\sqrt{cx+d}}dx$	Put $ax+b = A(cx+d)+B$. Find A and B by equating coefficients of like powers of x on both sides.
•	$\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$ or $\int \frac{1}{(ax^2+bx+c)\sqrt{cx+d}}dx$	Put $cx+d = t^2$
•	$\int \frac{1}{(ax+b)\sqrt{cx^2+dx+e}}dx$	Put $ax+b = \frac{1}{t}$
•	$\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$	Put $x = \frac{1}{t}$ and then $c+dt^2 = u^2$
•	$\int \frac{cx+d}{ax^2+bx+c}dx$ or $\int \frac{cx+d}{\sqrt{ax^2+bx+c}}dx$ or $\int (cx+d)\sqrt{ax^2+bx+c}\,dx$	$(cx+d) = A\frac{d}{dx}(ax^2+bx+c)+B$. Find A and B.
•	$\frac{a\sin x + b\cos x}{c\sin x + d\cos x}$	$a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$
•	$\frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f}$	$a\sin x + b\cos x + c = A\frac{d}{dx}(d\sin x + e\cos x + f) + B(d\sin x + e\cos x + f) + C$

- **Special Integrals**

(i) $\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$

(ii) $\int e^{kx}[kf(x)+f'(x)]\,dx = e^{kx} f(x) + C$

- **Some Important Substitutions**

	Expression as integrand	Substitution
1.	$a^2 + x^2$	$x = a \tan\theta$ or $a \cot\theta$
2.	$x^2 - a^2$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
3.	$a^2 - x^2$	$x = a \sin\theta$ or $a \cos\theta$
4.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
5.	$\sqrt{(x-a)(b-x)}$ or $\sqrt{\frac{x-a}{b-x}}$	$x = a\cos^2\theta + b\sin^2\theta$

- **Some Particular Integrals**

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$

(vi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C,\ |x|<a$

(vii) $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+C$

(viii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

(ix) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

(b) Integration by Partial Fractions

If $f(x)$ and $g(x)$ are two polynomials such that the degree of $f(x)$ is less than the degree of $g(x)$, then we can evaluate $\int \frac{f(x)}{g(x)}dx$ by decomposing $\frac{f(x)}{g(x)}$ into partial fractions.

	Form of a Rational Function	Form of the Partial Fractions
1.	$\frac{cx+d}{(x-a)(x-b)}$ $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$,
2.	$\frac{cx+d}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{cx+d}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
4.	$\frac{cx^2+dx+e}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
5.	$\frac{cx^2+dx+e}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
6.	$\frac{cx^2+dx+e}{(x-a)(x^2+bx+c)}$ where x^2+bx+c can not be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts

If u and v are two differentiable functions of x, then

$$\int (uv)\,dx = [u\cdot\int v\,dx] - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}dx$$

In order to choose 1st function, we take the letter which comes first in the word ILATE, where,

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

Note : If the second function is not given, we take it as 1.

DEFINITE INTEGRALS

Definition : A definite integral is denoted as $\int_a^b f(x)dx$, where a is the lower limit and b is the upper limit.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let $f(x)$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a)+f(a+h)+f(a+2h)+\ldots +\ldots+f\{a+(n-1)h\}]$$

where $h=\dfrac{b-a}{n}\to 0$ as $n\to\infty$

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

(i) **Area function :** The function $A(x)$ denotes the area function and is given by

$$A(x)=\int_a^x f(x)\,dx$$

(ii) **First Fundamental Theorem :** Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then, $A'(x) = f(x)$, for all $x \in [a, b]$

(iii) **Second Fundamental Theorem :** Let f be a continuous function defined on the closed interval $[a, b]$ and $\int f(x)\,dx = F(x)$.

Then, $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$

PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx=\int_a^b f(t)dt$

(ii) $\int_a^a f(x)dx=0$

(iii) $\int_a^b f(x)dx=-\int_b^a f(x)dx$

(iv) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

(v) $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

(vi) $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

(vii) $\int_{-a}^a f(x)dx=2\int_0^a f(x)dx,$ if $f(-x)=f(x)$

i.e; if $f(x)$ is even.

(viii) $\int_{-a}^a f(x)dx=0$ if $f(-x)=-f(x)$

i.e; if $f(x)$ is odd.

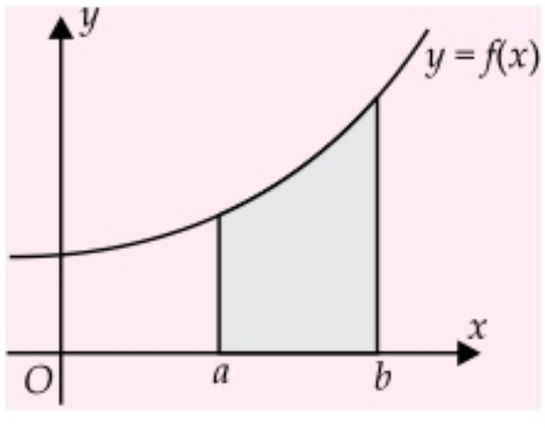
(ix) $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$

(x) $\int_0^{2a} f(x)dx\begin{cases}=2\int_0^a f(x)dx, \text{ if } f(x)=f(2a-x)\\ =0, \text{ if } f(2a-x)=-f(x)\end{cases}$

APPLICATION OF INTEGRALS

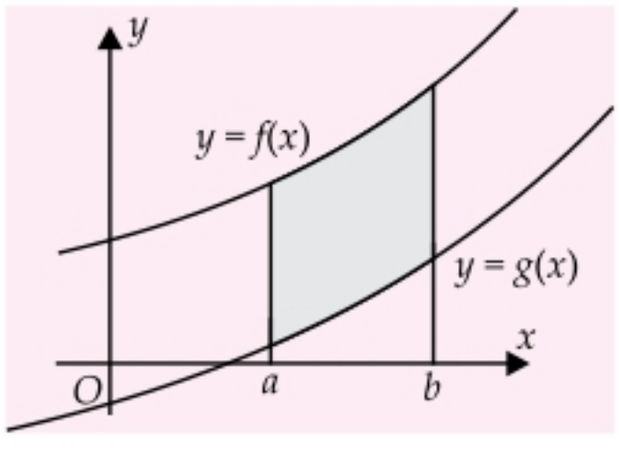
(i) The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is

$$A=\int_a^b y\,dx=\int_a^b |f(x)|\,dx$$

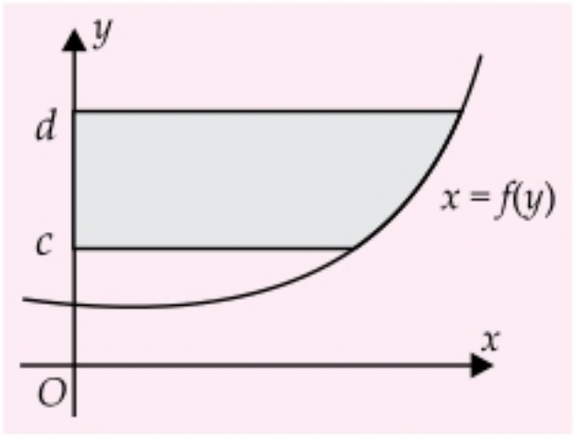


(ii) The area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A=\int_a^b |f(x)-g(x)|\,dx, \text{ where } f(x)\geq g(x) \text{ in } [a, b]$$



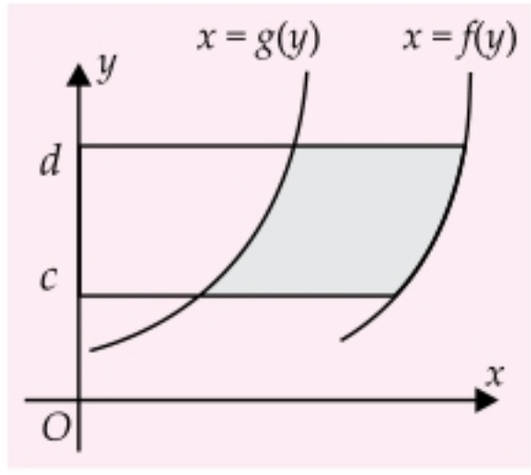
(iii) The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = c$ and $y = d$ is



$$A = \int_c^d x\,dy = \int_c^d |f(y)|\,dy$$

(iv) The area bounded by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is

$$A = \int_c^d |f(y) - g(y)|\,dy$$



Note : If $f(x) \geq g(x)$ in $[c, e]$ and $f(x) \leq g(x)$ in $[e, d]$ where $c < e < d$, then area bounded by the curves is,

$$\int_c^e [f(x) - g(x)]\,dx + \int_e^d [g(x) - f(x)]\,dx$$

DIFFERENTIAL EQUATIONS

Definition : An equation involving derivative (derivatives) of dependent variable with respect to independent variable is called the differential equation.

- A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an **ordinary differential equation** and a differential equation involving derivatives with respect to more than one independent variables is called a **partial differential equation.**

ORDER OF A DIFFERENTIAL EQUATION

It is the order of the highest order derivative in the given differential equation.

The order of a differential equation is always positive.

DEGREE OF A DIFFERENTIAL EQUATION

The highest index (power) of the highest order derivative involved in the given differential equation.

Note: (i) Degree (if defined) of a differential equation is always a positive integer.

(ii) The degree of a differential equation which is not a polynomial equation in derivatives is not defined.

SOLUTION OF DIFFERENTIAL EQUATION

A relation between involved variables, which satisfy the given differential equation, is called its solution.

Types of Solution

(i) **General Solution :** The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

(ii) **Particular Solution :** Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

FORMATION OF DIFFERENTIAL EQUATION

Here we differentiate the given equation as many times as the number of arbitrary constants and then eliminate the arbitrary constants from them.

METHODS OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

(i) If the equation is $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx + C$.

(ii) **Variable Separable :** If the equation is $\frac{dy}{dx} = f(x) \cdot g(y)$, then $\int \frac{dy}{g(y)} = \int f(x) + C$.

(iii) **Reducible to Variable separable :** If the equation is $\frac{dy}{dx} = f(ax + by + c)$, then put $ax + by + c = z$.

(iv) **Homogeneous Equation :** If the equation is $\frac{dy}{dx}=\frac{f(x,\ y)}{g(x,\ y)}$, where $f(x, y)$, $g(x, y)$ are homogeneous functions of the same degree in x and y, then put $y = vx$.

(v) **Linear Equation :** If the equation is $\frac{dy}{dx}+Py=Q$, where P, Q are functions of x, then

$ye^{\int Pdx}-\int Qe^{\int Pdx}dx+C$ where $e^{\int Pdx}$ is the integrating factor (I.F.).

Or

If the equation is $\frac{dx}{dy}+Px=Q$, where P, Q are functions of y, then

$xe^{\int Pdy}=Qe^{\int Pdy}dy+C$ where $e^{\int Pdy}$ is the integrating factor.

VERY SHORT ANSWER TYPE (1 MARK)

1. Evaluate $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx.$

2. Evaluate $\int_0^{\pi/4} \tan^2 x\ dx.$

3. Find the differential equation for the family of all concentric circles centred at the origin and having different radii.

4. Determine the degree and order of the differential equation $\frac{d^5y}{dx^5}+e^{dy/dx}+y^2=0$

5. Evaluate $\int_2^3 \frac{x}{x^2+1}dx.$

6. Solve : $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}.$

SHORT ANSWER TYPE (4 MARKS)

7. Evaluate : $\int \frac{x^3\ \sin^{-1}\ x^4}{\sqrt{1-x^4}}\ dx$

8. Using integration, find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 2$ and $x = 4$.

9. Solve the differential equation

$$\frac{dy}{dx}=\frac{e^x(\sin^2 x+\sin 2x)}{y(2\log y+1)}$$

10. Evaluate : $\int \frac{2x+5}{\sqrt{x^2+3x+2}}\ dx$

11. Prove that

$$\int_0^{\pi/2} (2\ \log\ \sin x-\log\ \sin 2x)\ \ dx=-\frac{\pi}{2}(\log\ 2).$$

12. Resolve $\frac{16}{(x-2)(x+2)^2}$ into partial fractions.

13. Evaluate $\int x^2 \sin^{-1} x\ dx$

14. Solve $(1 + y^2)\ dx = (\tan^{-1} y - x)\ dy$

LONG ANSWER TYPE (6 MARKS)

15. Find the area of the region represented by $\{(x, y) : x^2 \le y \le |\ x\ |\}$.

16. Evaluate $\int_a^b \sin x\ dx$ as limit of sum.

17. Evaluate $\int \frac{(x^2+1)e^x}{(x+1)^2}\ dx.$

18. Solve : $x^2y\ dx - (x^3 + y^3)\ dy = 0$

19. Find the solution of the differential equation

$y-x\frac{dy}{dx}=2\left(1+x^2\frac{dy}{dx}\right)$ given that $y = 1$ when $x = 1$.

20. Evaluate $\int_0^{\pi} \frac{x\ \tan x}{\sec x+\tan x}\ dx$

SOLUTIONS

1. $\int \tan^{-1}\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$

$$=\int \tan^{-1}\left(\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right)dx=\int \tan^{-1}(\tan x)dx$$

$$=\int x\ dx=\frac{1}{2}x^2+C.$$

2. $\int_0^{\pi/4} \tan^2 x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\, dx = \left[\tan x - x\right]_0^{\pi/4}$

$$= \left(1 - \frac{\pi}{4}\right)$$

3. The equation of the given family of circles is
$x^2 + y^2 = r^2$, r being a parameter ...(i)
On differentiating (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$$

is the required differential equation.

4. The order of the differential equation is 5 and its degree is not defined.

5. $\int_2^3 \frac{x}{x^2+1}\, dx = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\, dx$

$$= \frac{1}{2}\left[\log|x^2+1|\right]_2^3 = \frac{1}{2}\left\{\log(3^2+1) - \log(2^2+1)\right\}$$

$$= \frac{1}{2}(\log 10 - \log 5) = \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2.$$

6. We have, $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2) \Rightarrow e^y\, dy = (e^x + x^2)\, dx$$

Integrating both sides, we get

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$, is the required solution.

7. Let $I = \int \frac{x^3 \sin^{-1} x^4}{\sqrt{1-x^4}}\, dx$

Substitute $x^4 = t \Rightarrow 4x^3\, dx = dt \Rightarrow x^3\, dx = \frac{1}{4}\, dt$

$$\therefore \quad I = \frac{1}{4}\int \frac{\sin^{-1} t}{\sqrt{1-t}}\, dt = \frac{1}{4}\int (\sin^{-1} t)(1-t)^{-1/2}\, dt$$

$$\therefore\ I = \frac{1}{4}\left[\sin^{-1} t \cdot \frac{(1-t)^{1/2}}{\frac{1}{2}\cdot(-1)} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{(1-t)^{1/2}}{\frac{1}{2}\cdot(-1)}\, dt\right]$$

$$= -\frac{1}{2}\sqrt{1-t}\sin^{-1} t + \frac{1}{2}\int \frac{\sqrt{1-t}}{\sqrt{1-t^2}}\, dt$$

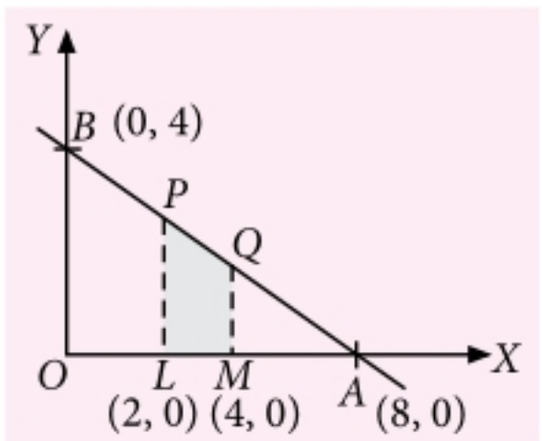
$$= -\frac{1}{2}\sqrt{1-t}\sin^{-1} t + \frac{1}{2}\int (1+t)^{-1/2} dt$$

$$= -\frac{1}{2}\sqrt{1-t}\sin^{-1} t + \sqrt{1+t} + C$$

$$= -\frac{1}{2}\sqrt{1-x^4}\sin^{-1} x^4 + \sqrt{1+x^4} + C$$

8. The given line AB is

$$2y + x = 8 \Rightarrow y = 4 - \frac{x}{2} \qquad \text{...(i)}$$



$\therefore$ Required area = area $PLMQP$
= area between the line $y = 4 - \frac{x}{2}$, and the x-axis between $x = 2$ and $x = 4$.

$$= \int_2^4 y_{AB}\, dx = \int_2^4 \left(4 - \frac{1}{2}x\right) dx$$

$$= \left[4x - \frac{1}{4}x^2\right]_2^4 = (12 - 7) \text{ sq. units}$$

= 5 sq. units.

9. We have, $\frac{dy}{dx} = \frac{e^x(\sin^2 x + \sin 2x)}{y(2\log y + 1)}$

$\Rightarrow \quad y(2\log y + 1)\, dy = e^x(\sin^2 x + \sin 2x)\, dx$

Integrating both sides, we get

$$2\int \underset{\text{II}}{y}\ \underset{\text{I}}{(\log y)}\, dy + \int y\, dy = \int e^x(\sin^2 x + \sin 2x)\, dx$$

$$\Rightarrow \quad 2\left[(\log y).\frac{y^2}{2} - \int \frac{1}{y}.\frac{y^2}{2}\, dy\right] + \frac{1}{2}y^2 = \int e^x(\sin^2 x + \sin 2x)\, dx$$

$$\Rightarrow \quad y^2(\log y) - \int y\, dy + \frac{1}{2}y^2 = \int e^x(\sin^2 x + \sin 2x)\, dx$$

$\Rightarrow \quad y^2(\log y) = e^x \sin^2 x + C$ is the required solution.

10. $\int \frac{2x+5}{\sqrt{x^2+3x+2}}dx = \int \frac{(2x+3)+2}{\sqrt{x^2+3x+2}}dx$

$= \int (x^2+3x+2)^{-1/2}(2x+3)dx + 2\int \frac{dx}{\sqrt{x^2+3x+2}}$

$= \frac{(x^2+3x+2)^{\frac{1}{2}}}{1/2} + 2\int \frac{dx}{\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= 2\sqrt{x^2+3x+2} + 2\log\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + C$

$= 2\sqrt{x^2+3x+2} + 2\log\left|x+\frac{3}{2}+\sqrt{x^2+3x+2}\right| + C$

11. Let $I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx$ ...(i)

Then,

$I = \int_0^{\pi/2}\left[2\log\sin\left(\frac{\pi}{2}-x\right) - \log\sin 2\left(\frac{\pi}{2}-x\right)\right]dx$

$\Rightarrow I = \int_0^{\pi/2} (2\log\cos x - \log\sin 2x)\,dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} [2(\log\sin x + \log\cos x) - 2\log\sin 2x]\,dx$

or $I = \int_0^{\pi/2} [\log(\sin x\cos x) - \log\sin 2x]dx$

$= \int_0^{\pi/2}\left[\log\left(\frac{\sin 2x}{2}\right) - \log\sin 2x\right]dx$

$= \int_0^{\pi/2} (\log\sin 2x - \log 2 - \log\sin 2x)\,dx$

$= -\log 2\cdot\int_0^{\pi/2} dx = (-\log 2)\cdot[x]_0^{\pi/2}$

$= -\frac{\pi}{2}(\log 2).$

12. Let $\frac{16}{(x-2)(x+2)^2} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

or $\frac{16}{(x-2)(x+2)^2}$

$= \frac{A(x+2)^2 + B(x-2)(x+2) + C(x-2)}{(x-2)(x+2)^2}$

$\therefore \quad 16 \equiv A(x+2)^2 + B(x-2)(x+2) + C(x-2)$ ...(i)

or $16 \equiv (A+B)x^2 + (4A+C)x + (4A-4B-2C)$...(ii)

Putting $x = 2$ in (i), we get $A = 1$.

Putting $x = -2$ in (i), we get $C = -4$.

Comparing coefficients of x^2 on both sides of (ii), we get

$A + B = 0$ or $B = -A = -1$.

Thus $A = 1$, $B = -1$ and $C = -4$.

$\therefore \quad \frac{16}{(x-2)(x+2)^2} = \left[\frac{1}{(x-2)} - \frac{1}{(x+2)} - \frac{4}{(x+2)^2}\right].$

13. Integrating by parts, taking $\sin^{-1}x$ as the first function, we get

$\int x^2\sin^{-1}x\,dx = (\sin^{-1}x)\cdot\frac{x^3}{3} - \int \frac{1}{\sqrt{1-x^2}}\cdot\frac{x^3}{3}dx$

$= \frac{x^3(\sin^{-1}x)}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx$

$= \frac{x^3(\sin^{-1}x)}{3} + \frac{1}{3}\int \frac{t(1-t^2)}{t}dt,$ where $(1-x^2) = t^2$

$= \frac{x^3(\sin^{-1}x)}{3} + \frac{1}{3}\int dt - \frac{1}{3}\int t^2\,dt$

$= \frac{x^3(\sin^{-1}x)}{3} + \frac{1}{3}t - \frac{1}{9}t^3 + C$

$= \frac{x^3(\sin^{-1}x)}{3} + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C$

14. The given differential equation may be written as $\frac{dx}{dy} = \frac{\tan^{-1}y - x}{(1+y^2)}$

$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{(1+y^2)}\cdot x = \frac{\tan^{-1}y}{(1+y^2)}$

This is of the form $\frac{dx}{dy} + Px = Q$, where

$P = \frac{1}{(1+y^2)}$ and $Q = \frac{\tan^{-1}y}{(1+y^2)}$.

Clearly, the given equation is linear.

$\therefore \quad$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$.

So, the required solution is

$$x \times \text{I.F.} = \int \{Q \times \text{I.F.}\}\, dy + C$$

$$\Rightarrow \quad x \times e^{\tan^{-1} y} = \int \left\{ \frac{\tan^{-1} y}{(1+y^2)} \times e^{\tan^{-1} y} \right\} dy + C$$

$$= \int (te^t)\, dt + C, \text{ where } \tan^{-1} y = t$$

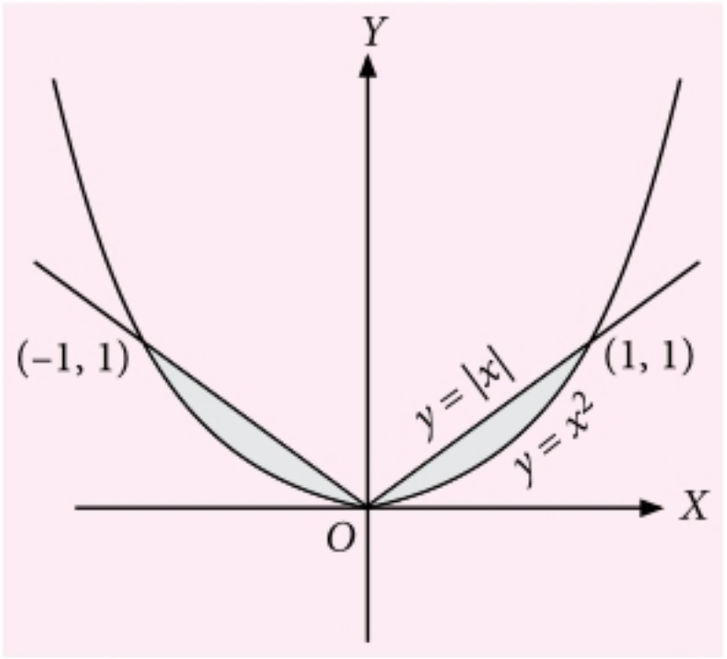
$$= te^t - \int 1 \cdot e^t\, dt + C = (t-1)e^t + C$$

$$= (\tan^{-1} y - 1)e^{\tan^{-1} y} + C$$

15. Consider the curves $y = x^2$ ...(1)

and $y = |x|$ ...(2)

The required area is enclosed between these two curves and is shown shaded in the figure.



Hence, the two curves meet at the points (–1, 1), (0, 0) and (1, 1). As the area to be found is symmetrical about y-axis.

$\therefore$ Required area = 2 × (area shown shaded in the first quadrant)

$$= 2\int_0^1 (|x| - x^2)dx = 2\left[\int_0^1 |x|\,dx - \int_0^1 x^2 dx\right]$$

$$= 2\left[\int_0^1 x\, dx - \int_0^1 x^2\, dx\right] = 2\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right)$$

$$= 2\left\{\left(\frac{1}{2} - 0\right) - \left(\frac{1}{3} - 0\right)\right\} = \frac{1}{3} \text{ square units.}$$

16. We have,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots$$

$$\ldots + f(a + (n-1)h)], \text{ where } h = \frac{b-a}{n}.$$

Here, $f(x) = \sin x$

$$\therefore \quad I = \int_a^b \sin x\, dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h[\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin(a + (n-1)h)]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\frac{\sin\left(a + (n-1)\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\frac{\sin\left(a + \frac{nh}{2} - \frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\frac{\sin\left(a + \frac{b-a}{2} - \frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)}{\sin\frac{h}{2}}\right]$$

$$[\because \quad nh = b - a]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right) \times$$

$$\lim_{h \to 0} 2 \sin\left(\frac{a+b}{2} - \frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$\Rightarrow \quad I = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$\Rightarrow \quad I = \cos a - \cos b$$

17. We have

$$I = \int e^x \cdot \frac{(x^2+1)}{(x+1)^2}\, dx = \int e^x \cdot \left\{\frac{(x+1)^2 - 2x}{(x+1)^2}\right\} dx$$

$$= \int e^x \cdot \left\{1 - \frac{2x}{(x+1)^2}\right\} dx = \int e^x\, dx - 2\int e^x \cdot \frac{x}{(x+1)^2}\, dx$$

$$= e^x - 2\int e^x \cdot \frac{\{(x+1)-1\}}{(x+1)^2}\, dx$$

$$= e^x - 2\int e^x \cdot \left\{\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right\} dx$$

$$= e^x - 2\int e^x \{f(x) + f'(x)\} dx, \text{ where } f(x) = \frac{1}{x+1}$$

and $f'(x) = \dfrac{-1}{(x+1)^2}$

$$= e^x - 2e^x \cdot f(x) + C = e^x - 2e^x \cdot \frac{1}{(x+1)} + C$$

$$= e^x \cdot \left\{1 - \frac{2}{(x+1)}\right\} + C = e^x\left(\frac{x-1}{x+1}\right) + C.$$

18. The given differential equation is

$x^2y\,dx - (x^3 + y^3)\,dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3} \qquad ...(i)$$

Since each of the function x^2y and $x^3 + y^3$ is a homogeneous function of degree 3, so the given differential equation is homogeneous.

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - v - v^4}{1+v^3} \quad \Rightarrow \quad x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}dv = -\frac{dx}{x} \quad \Rightarrow \left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\frac{dx}{x},$$

Integrating both sides, we get

$$\Rightarrow \quad \frac{v^{-3}}{-3} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad -\frac{1}{3v^3} + \log|v| + \log|x| = C$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log|y| = C, \text{ is the required solution.}$$

19. The given differential equation is

$$y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (y-2)\,dx = x(2x+1)\,dy$$

$$\Rightarrow \quad \frac{dx}{x(2x+1)} = \frac{dy}{(y-2)},$$

Integrating both sides, we get

$$\Rightarrow \quad \int\left(\frac{1}{x} - \frac{2}{2x+1}\right)dx = \int\frac{1}{(y-2)}\,dy$$

$$\Rightarrow \quad \log|x| - \log|2x+1| = \log|y-2| + \log\ C$$

$$\Rightarrow \quad \log\left|\frac{x}{2x+1}\right| = \log|y-2| + \log\ C$$

$$\Rightarrow \quad \log\left|\frac{x}{2x+1}\right| - \log|y-2| = \log C$$

$$\Rightarrow \quad \left|\frac{x}{(2x+1)\ (y-2)}\right| = C \qquad ...(i)$$

It is given that $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (i), we get $C = \frac{1}{3}$

Putting $C = \frac{1}{3}$ in (i), we get

$$\left|\frac{x}{(2x+1)(y-2)}\right| = \frac{1}{3}$$

$$\Rightarrow \quad y - 2 = \pm\frac{3x}{2x+1} \Rightarrow y = 2 \pm \frac{3x}{2x+1}$$

But, $y = 2 + \frac{3x}{2x+1}$ is not satisfied by $y(1) = 1$.

Hence, $y = 2 - \frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$ is the required solution.

20. Let $I = \int_0^{\pi} \frac{x\ \tan x}{\sec x + \tan x}\,dx = \int_0^{\pi} \frac{x\ \sin x}{1+\sin x}dx \qquad ...(i)$

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin\ (\pi - x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi\ \sin x - x\ \sin x}{1 + \sin\ x}\,dx \qquad ...(ii)$$

Adding (i) and (ii), we get $2I = \int_0^{\pi} \frac{\pi\ \sin x}{1+\sin\ x}\,dx$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{\sin x(1 - \sin x)}{1 - \sin^2\ x}\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2\ x}\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} (\tan\ x\ \sec\ x - \tan^2 x)\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} (\sec x\ \tan x - \sec^2 x + 1)\,dx$$

$$\Rightarrow \quad 2I = \pi\left[\sec x\ - \tan x + x\right]_0^{\pi}$$

$$\Rightarrow \quad 2I = \pi[\sec\pi - \tan\pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$$

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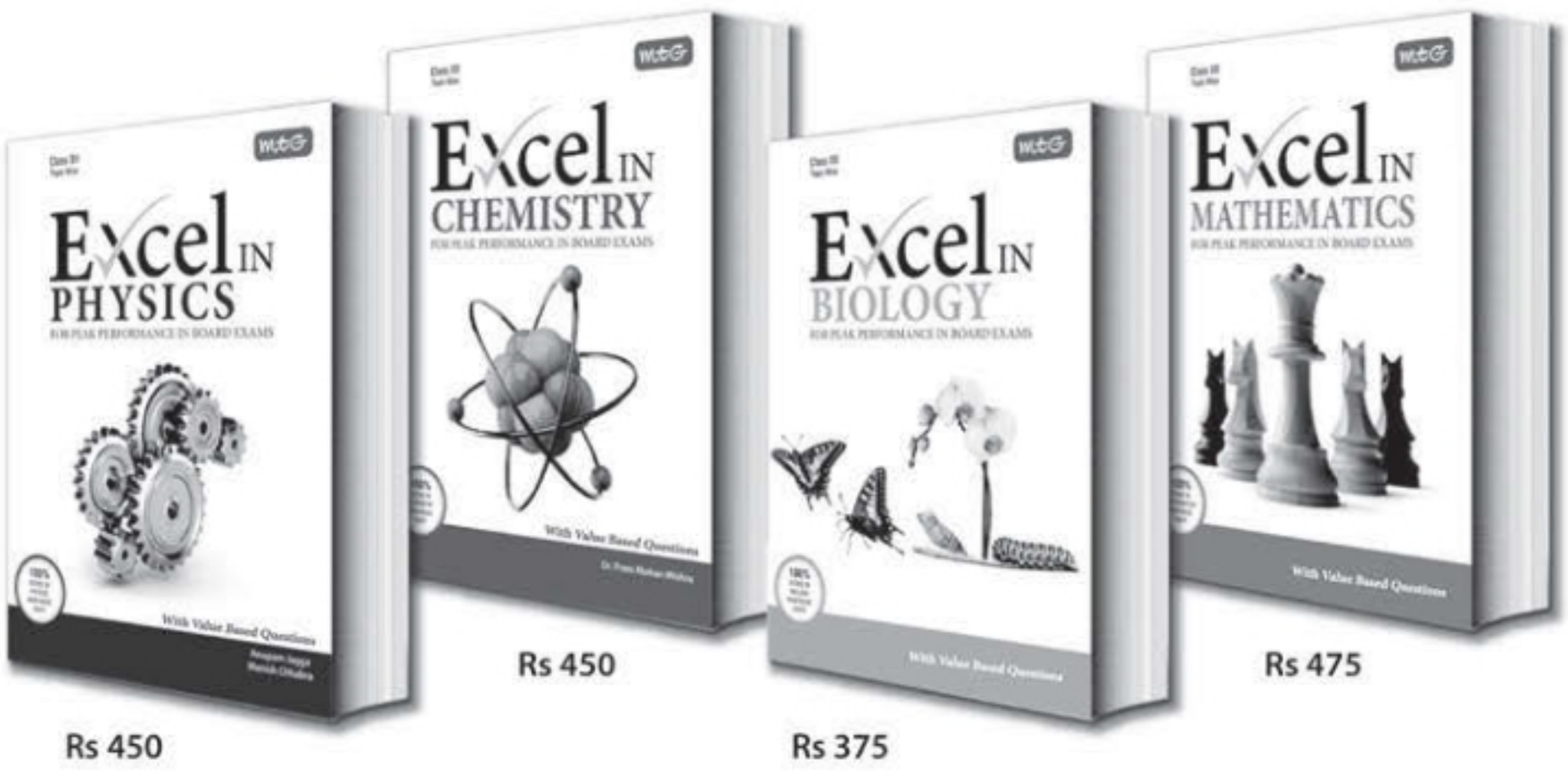

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MATHS MUSING

SOLUTION SET-142

1. **(c):** $f(x) = x - 1$ for $x = 2, 3, 4, 5$

$\therefore\ f(x) - x + 1 = (x-2)(x-3)(x-4)(x-5)$

$f(1) = 24,\ a + b + c + d = 23$

2. **(b) :** The number of 6-digit numbers is $\frac{6!}{2!3!} = 60$. In the unit place 1 occur 10 times, 2 occur 20 times and 3 occur 30 times.

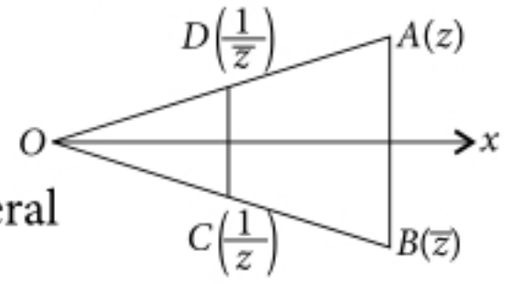
$\therefore\ S = 111111 \times (1 \times 10 + 2 \times 20 + 3 \times 30)$

$= 111111 \times 140 = 15555540$

Sum of the digits of $S = 30$

3. **(d) :** $z = x + iy,\ \bar{z} = x - iy$

$\frac{1}{z} = \frac{x-iy}{|z|^2}, \frac{1}{\bar{z}} = \frac{x+iy}{|z|^2}$



$\therefore$ Area of quadrilateral $ABCD$

$= $ Area of $\Delta OAB -$ Area of $\Delta OCD = xy - \frac{xy}{|z|^4}$

$= \frac{1}{4}\left|1 - \frac{1}{|z|^4}\right| |z^2 - \bar{z}^2|$

4. **(c):** With 3 zeroes on the diagonal

$$\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}$$

with 1 zero on the diagonal

$$\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix} \text{ etc. Total } = \binom{3}{1} + \binom{3}{1}\binom{3}{2} = 12$$

5. **(b) :** $(\sin A + \sin B)^2 + (\cos A + \cos B)^2 = 2$

$\Rightarrow \cos(B - A) = 0$

$\Rightarrow B = A + 90°,\ \cos A - \sin A = \frac{1}{\sqrt{2}}$

$\sin(45° - A) = \frac{1}{2} \Rightarrow A = 15°,\ A + B = 120°$

6. **(a, b, c) :** Two terms are odd. The expression is even

When n is odd, $a^n - b^n$ is divisible by $a - b$

When n is odd, $a^n + b^n$ is divisible by $a + b$.

If $a + b$ is even then $a^n + b^n$ is divisible by $\frac{a+b}{2}$.

$\therefore$ The given number is divisible by 2,

$27 = 2015 - 1988 = 1978 - 1951$ and

$37 = 2015 - 1978 = 1988 - 1951$

$\therefore$ It is divisible by $2 \times 27 \times 37 = 1998$,

$2015 + 1951 = 1988 + 1978 = 3966$

$\therefore$ Given number is divisible by 3966 and 1983

7. **(c):** The plane through L_1 is

$2x - 2y + 3z - 2 + \lambda(x - y + z + 1) = 0$

The plane through L_2 is

$x + 2y - z - 3 + \mu(3x - y + 2z - 1) = 0$

The two planes are same

$\therefore\ \frac{2+\lambda}{1+3\mu} = \frac{2+\lambda}{\mu-2} = \frac{3+\lambda}{2\mu-1} = \frac{2-\lambda}{3+\mu}$

$\Rightarrow \lambda = 5,\ \mu = -\frac{3}{2}$

$\therefore$ The plane is $7x - 7y + 8z + 3 = 0$

So, distance from origin is $\frac{3}{\sqrt{162}} = \frac{1}{3\sqrt{2}}$.

8. **(a):** $L_1 : \vec{r} = \vec{a} + s\vec{b}, L_2 : \vec{r} = \vec{c} + t\vec{d}$

$\vec{a} = 5\hat{i} + 4\hat{k}, \vec{c} = \frac{5}{7}\hat{i} + \frac{8}{7}\hat{j}$

$\vec{b} = \hat{i} + \hat{j},\ \vec{d} = 3\hat{i} - 5\hat{j} - 7\hat{k}$

$\vec{a} + s\vec{b} = \vec{c} + t\vec{d}$

$\Rightarrow\ s = \frac{5}{7} + 3t,\ 5 + s = \frac{8}{7} - 5t,\ h = -7t$

$t = -\frac{4}{7}, s = R - 1$

$P(-1, 4, 4),\ OR = \sqrt{33}$

9. **(3):** $16\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$

$= 9(1 + c)(c - 1)(3 - c)(3 + c)$

$= 9(16 - (c^2 - 5)^2) \le 9 \times 16$

$\Rightarrow\ \Delta \le 3$

10. **(c):** **P.** $f'(x) = \frac{1}{(1+x)^2} > 0$ $\therefore$ one-one

$f(x) \to 1$ as $x \to \infty$ $\therefore$ not onto

Q. $f'(x) = 1 + \frac{1}{x^2}$ $\therefore$ increase for $x > 0, x < 0$

$f(1) = f(-1) = 0$. $\therefore$ onto but not one-one.

R. $f'(x) = 1 - \frac{1}{x^2}$ not one-one, not onto

S. $f'(x) = 2 + \cos x > 0$, one-one and onto.

Solution Sender of Maths Musing

SET-141

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SET-142

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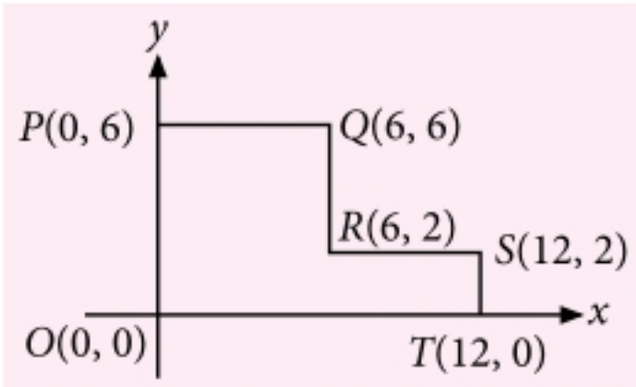
OLYMPIAD CORNER

CHALLENGING PROBLEMS FOR RMO, InMO, IMO & OTHER CONTESTS

1. In the diagram, two straight lines are to be drawn through $O(0, 0)$ so that the lines divide the figure $OPQRST$ into 3 pieces of equal area. Find the sum of the slopes of the lines.



2. Determine all linear functions $f(x) = ax + b$ such that if $g(x) = f^{-1}(x)$ for all values of x, then $f(x) - g(x) = 44$ for all values of x.
(Note : f^{-1} is the inverse function of f)

3. **(a)** Determine all pairs (a, b) of positive integers for which $a^3 + 2ab = 2013$.

(b) Determine all real values of x for which $\log_2(2^{x-1} + 3^{x+1}) = 2x - \log_2(3^x)$.

4. Integers can be written in bases other than the usual base 10. For example, the notation $(235)_7$ stands for the base 7 representation of the integer $2 \times 7^2 + 3 \times 7 + 5$ (which equals 124 in base 10). In general, if x, y and z are integers between 0 and $b - 1$, inclusive, then $(xyz)_b = xb^2 + yb + z$.
Find all triples (x, y, z) for which $(xyz)_{10} = 2(xyz)_7$, where each of x, y and z comes from the list 1, 2, 3, 4, 5, 6.

5. Vernon starts with a first number n with $0 < n < 1$. Vernon enters the first number into a machine to produce a second number. He then enters the second number back into the machine to produce a third number. Vernon continually enters the result back into the machine giving a chain of numbers.

When Vernon enters a number x into the machine,

- if $x \le \frac{1}{2}$, the machine outputs $2x$, and
- if $x > \frac{1}{2}$, the machine outputs $2(1 - x)$.

If the machine ever produces the number 1, Vernon stops the process.

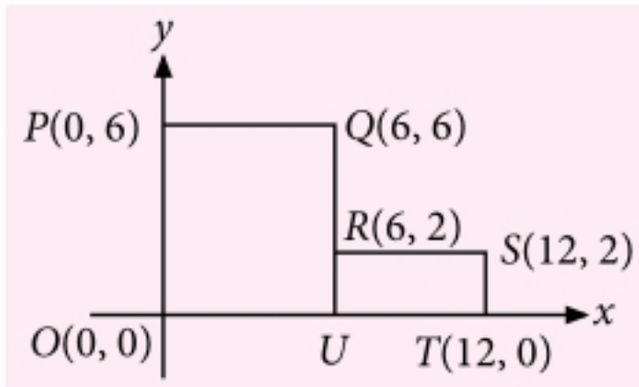
(a) A chain starts with $\frac{3}{11}$. This gives $\frac{3}{11} \to \frac{6}{11} \to \frac{10}{11} \to \frac{2}{11} \to \ldots$. What are the next four numbers in this chain?

(b) Vernon enters a number x with $0 < x < 1$ into the machine and the machine produces the number x. Determine the value of x.

(c) The fourth number in a chain is 1. Determine all of the possible values of the first number in this chain.

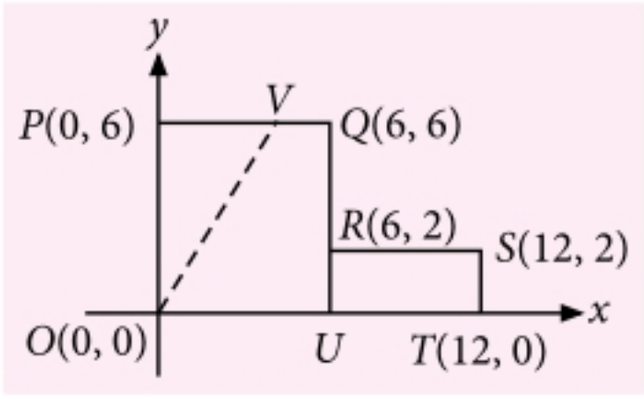
SOLUTIONS

1. Extend QR downwards to meet the x-axis at $U(6, 0)$.



The area of figure $OPQRST$ equals the sum of the areas of square $OPQU$ (which has side length 6, so area 36) and rectangle $RSTU$ (which has height 2 and width 6, so area 12).
Thus, the area of figure $OPQRST$ is 48. If we are to divide the figure into three pieces of equal area, then each piece has area 16.
Let V be the first point on the perimeter

(measured clockwise from P) so that the line through O and V cuts off an area of 16.
Note that the area of ΔOPQ is half of the area of square $OPQU$, or 18, so V is to the left of Q on PQ.
Thus, V has coordinates $(v, 6)$ for some number v.



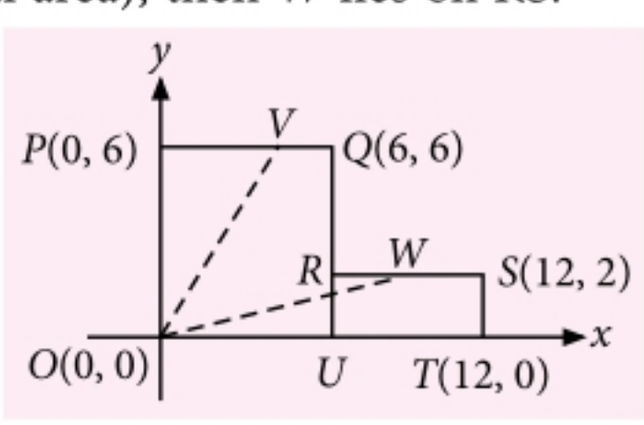
Consider ΔOPV having base OP of length 6 and height PV of length v.
Since the area of ΔOPV is 16, then

$\frac{1}{2}(6)(v) = 16$ or $3v = 16$ or $v = \frac{16}{3}$.

Therefore, the slope of OV is $\frac{6}{\frac{16}{3}} = \frac{9}{8}$.

Let W be the second desired point.

Since the area of ΔOTS is $\frac{1}{2}(OT)(TS)$ $= \frac{1}{2}(12)(2) = 12$ (less than $\frac{1}{3}$ of the total area) and the area of trapezoid $ORST$ is $\frac{1}{2}(RS + OT)$ $(ST) = \frac{1}{2}(6 + 12)(2) = 18$ (more than $\frac{1}{3}$ of the total area), then W lies on RS.



Suppose that W has coordinates $(w, 2)$ for some number w.
We want the area of trapezoid $WSTO$ to be 16.

Therefore, $\frac{1}{2}(WS + OT)(ST) = 16$

$\Rightarrow \frac{1}{2}(12 - w + 12)(2) = 16$

$\Rightarrow 24 - w = 16 \Rightarrow w = 8$

Thus, the coordinates of W are (8, 2), and so the slope of OW is $\frac{2}{8} = \frac{1}{4}$.

Thus, the sum of the two required slopes is $\frac{9}{8} + \frac{1}{4} = \frac{11}{8}$.

2. Since $f(x) = ax + b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ and to obtain $y = ax + b$. We then interchange x and y to obtain $x = ay + b$ which we solve for y to obtain $ay = x - b$ or $y = \frac{x}{a} - \frac{b}{a}$.

Therefore, $f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$.

Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)
Therefore, the equation $f(x) - g(x) = 44$ becomes $(ax + b) - \left(\frac{x}{a} - \frac{b}{a}\right) = 44$ or $\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 44 = 0x + 44$, and this equation is true for all x.
Since the equation

$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 0x + 44$$

is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.

Therefore, $a - \frac{1}{a} = 0$ and $b + \frac{b}{a} = 44$.

From the first of these equations, we obtain

$a = \frac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.

If $a = 1$, the equation $b + \frac{b}{a} = 44$ becomes $b + b = 44$, which gives $b = 22$.

If $a = -1$, the equation $b + \frac{b}{a} = 44$ becomes $b - b = 44$, which is not possible.
Therefore, we must have $a = 1$ and $b = 22$, and so $f(x) = x + 22$.

3. **(a)** First, we factor the left side of the given equation to obtain $a(a^2 + 2b) = 2013$.

Next, we factor the integer 2013 as $2013 = 3 \times 671 = 3 \times 11 \times 61$. Note that each of 3, 11 and 61 is prime, so we can factor 2013 no further (We can find the factors of 3 and 11 using tests for divisibility by 3 and 11, or by systematic trial and error.)

Since $2013 = 3 \times 11 \times 61$, then the positive divisors of 2013 are

1, 3, 11, 33, 61, 183, 671, 2013

Since a and b are positive integers, then a and $a^2 + 2b$ are both positive integers.

Since a and b are positive integers, then $a^2 \geq a$ and $2b > 0$, so $a^2 + 2b > a$.

Since $a(a^2 + 2b) = 2013$, then a and $a^2 + 2b$ must be a divisor pair of 2013 (that is, a pair of positive integers whose product is 2013) with $a < a^2 + 2b$.

We make a table of the possibilities:

a	$a^2 + 2b$	$2b$	b
1	2013	2012	1006
3	671	662	331
11	183	62	31
33	61	−1028	N/A

Note that the last case is not possible, since b must be positive.

Therefore, the three pairs of positive integers that satisfy the equation are (1, 1006), (3, 331), (11, 31).

(We can verify by substitution that each is a solution of the original equation.)

(b) We successively manipulate the given equation to produce equivalent equations :

$\log_2(2^{x-1} + 3^{x+1}) = 2x - \log_2(3^x)$

$\log_2(2^{x-1} + 3^{x+1}) + \log_2(3^x) = 2x$

$\log_2((2^{x-1} + 3^{x+1})3^x) = 2x$

(using $\log_2 A + \log_2 B = \log_2 AB$)

$(2^{x-1} + 3^{x+1})3^x = 2^{2x}$

(exponentiating both sides)

$2^{-1}2^x3^x + 3^13^x3^x = 2^{2x}$

$\frac{1}{2} \cdot 2^x3^x + 3 \cdot 3^{2x} = 2^{2x}$

$2^x3^x + 6 \cdot 3^{2x} = 2 \cdot 2^{2x}$ (multiplying by 2)

$2^x3^x2^{-2x} + 6 \cdot 3^{2x}2^{-2x} = 2$

(dividing both sides by $2^{2x} \neq 0$)

$2^{-x}3^x + 6 \cdot 3^{2x} \cdot 2^{-2x} = 2$

$\left(\frac{3}{2}\right)^x + 6\left(\frac{3}{2}\right)^{2x} = 2$

Next, we make the substitution $t = \left(\frac{3}{2}\right)^x$, noting that $\left(\frac{3}{2}\right)^{2x} = \left(\left(\frac{3}{2}\right)^x\right)^2 = t^2$. Thus, we obtain the equivalent equations

$t + 6t^2 = 2$

$6t^2 + t - 2 = 0$

$(3t + 2)(2t - 1) = 0$

Therefore, $t = -\frac{2}{3}$ or $t = \frac{1}{2}$.

Since $t = \left(\frac{3}{2}\right)^x > 0$, then we must have $t = \left(\frac{3}{2}\right)^x = \frac{1}{2}$.

Thus,

$$x = \log_{3/2}(1/2) = \frac{\log(1/2)}{\log(3/2)} = \frac{\log 1 - \log 2}{\log 3 - \log 2}$$
$$= \frac{-\log 2}{\log 3 - \log 2} = \frac{\log 2}{\log 2 - \log 3}$$

4. Since we are told that $(xyz)_b = xb^2 + yb + z$, then $(xyz)_{10} = 10^2x + 10y + z = 100x + 10y + z$ and $(xyz)_7 = 7^2x + 7y + z = 49x + 7y + z$.

From the given information

$$\begin{aligned} (xyz)_{10} &= 2(xyz)_7 \\ 100x + 10y + z &= 2(49x + 7y + z) \\ 100x + 10y + z &= 98x + 14y + 2z \\ 2x &= 4y + z \end{aligned}$$

Since the left side of this equation is an even integer ($2x$), then the right side must also be an even integer.

Since $4y$ is an even integer, then for $4y + z$ to be an even integer, it must be the case that z is an even integer.

This gives us three possibilities: $z = 2$, $z = 4$ and $z = 6$.

Case 1 : $z = 2$

Here, $2x = 4y + 2$ or $x = 2y + 1$.

We try the possible values for y :

- $y = 1$ gives $x = 2(1) + 1 = 3$; this gives the triple $(x, y, z) = (3, 1, 2)$
- $y = 2$ gives $x = 2(2) + 1 = 5$; this gives the triple $(x, y, z) = (5, 2, 2)$
- If y is at least 3, then $x = 2y + 1$ is at least 7, which is impossible.

Therefore, there are two triples that work when $z = 2$.

Case 2: $z = 4$

Here, $2x = 4y + 4$ or $x = 2y + 2$.

We try the possible values for y :

- $y = 1$ gives $x = 2(1) + 2 = 4$; this gives the triple $(x, y, z) = (4, 1, 4)$
- $y = 2$ gives $x = 2(2) + 2 = 6$; this gives the triple $(x, y, z) = (6, 2, 4)$
- If y is at least 3, then $x = 2y + 2$ is at least 8, which is impossible.

Therefore, there are two triples that work when $z = 4$.

Case 3: $z = 6$

Here, $2x = 4y + 6$ or $x = 2y + 3$.

We try the possible values for y :

- $y = 1$ gives $x = 2(1) + 3 = 5$; this gives the triple $(x, y, z) = (5, 1, 6)$
- If y is at least 2, then $x = 2y + 3$ is at least 7, which is impossible.

Therefore, there is one triple that works when $z = 6$.

Finally, the triples that satisfy the equation are $(x, y, z) = (3, 1, 2), (5, 2, 2), (4, 1, 4), (6, 2, 4), (5, 1, 6)$.

5. **(a)** Since the fourth number in the chain is $\frac{2}{11}$ and $\frac{2}{11}$ is less than $\frac{1}{2}$, then the fifth number is $2\left(\frac{2}{11}\right) = \frac{4}{11}$.

Since the fifth number in the chain is $\frac{4}{11}$ and $\frac{4}{11}$ is less than $\frac{1}{2}$, then the sixth number is $2\left(\frac{4}{11}\right) = \frac{8}{11}$.

Since the sixth number in the chain is $\frac{8}{11}$ and $\frac{8}{11}$ is larger than $\frac{1}{2}$, then the seventh number is $2\left(1 - \frac{8}{11}\right) = 2\left(\frac{3}{11}\right) = \frac{6}{11}$.

Since the seventh number in the chain is $\frac{6}{11}$ and $\frac{6}{11}$ is larger than $\frac{1}{2}$, then the eighth number is $2\left(1 - \frac{6}{11}\right) = 2\left(\frac{5}{11}\right) = \frac{10}{11}$.

Therefore, the next four numbers in the chain are $\frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}$.

(b) Therefore are two possibilities :

$x \le \frac{1}{2}$ and $x > \frac{1}{2}$.

If $x \le \frac{1}{2}$ and x is entered into the machine, then the machine outputs $2x$.

Since the input is to be the same as the output, then $x = 2x$ or $x = 0$.

This is impossible since $x > 0$.

If $x > \frac{1}{2}$ and x is entered into the machine, then the machine outputs $2(1 - x)$.

Since the input is to be the same as the output, then $x = 2(1 - x)$ or $x = 2 - 2x$.

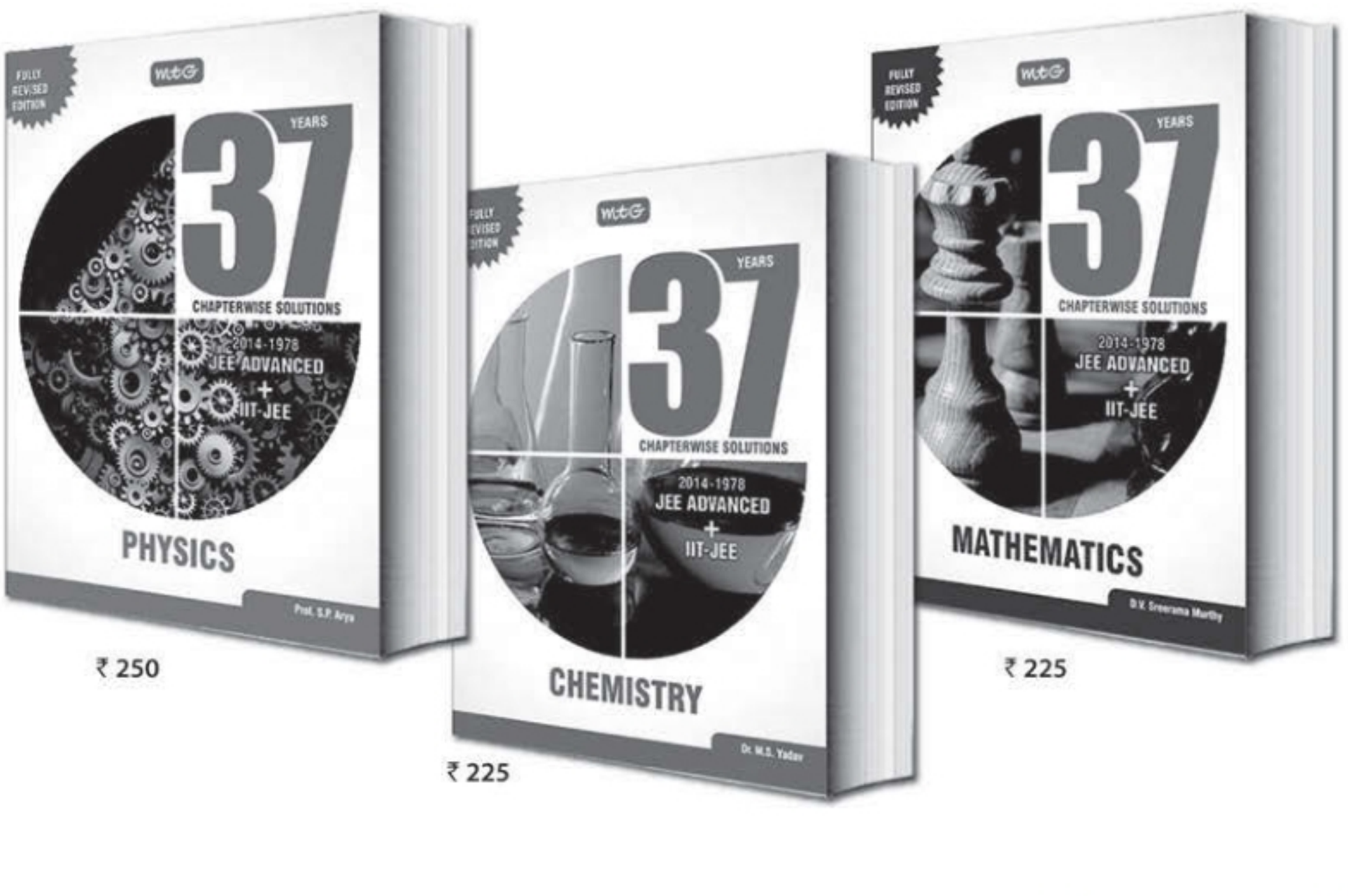
Thus, $3x = 2$ and so $x = \frac{2}{3}$, which satisfies the restrictions.

Therefore, if x is entered into the machine and x is produced, then $x = \frac{2}{3}$.

(c) Suppose that the chain is a, b, c, 1.

In this part, we have to produce the chain

Mad about rehearsing?



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backwards (that is, we have to determine c, b and a).

We make the following general observation: Suppose that x is entered into the machine and y is produced.

If $x \le \frac{1}{2}$, then $y = 2x$. Solving for x in terms of y, we obtain $x = \frac{1}{2}y$. (This gives us the input in terms of the output)

If $x > \frac{1}{2}$, then $y = 2(1 - x)$. Solving for x in terms of y, we obtain $y = 2 - 2x$ or $2x = 2 - y$ or $x = \frac{1}{2}(2 - y)$.

Since the third number in the chain is c and the fourth is 1, then from above,

$c = \frac{1}{2}(1) = \frac{1}{2}$ or $c = \frac{1}{2}(2 - 1) = \frac{1}{2}$.

In either case, the chain is a, b, $\frac{1}{2}$, 1.

Since the second number in the chain is b and the third is $\frac{1}{2}$, then from above,

$b = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ or $b = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$.

Therefore, the chain is either a, $\frac{1}{4}$, $\frac{1}{2}$, 1 or a, $\frac{3}{4}$, $\frac{1}{2}$, 1. If the chain is a, $\frac{1}{4}$, $\frac{1}{2}$, 1, then the first number is a and the second number is $\frac{1}{4}$.

Thus, either $a = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$ or

$a = \frac{1}{2}\left(2 - \frac{1}{4}\right) = \frac{1}{2}\left(\frac{7}{4}\right) = \frac{7}{8}$.

If the chain is a, $\frac{3}{4}$, $\frac{1}{2}$, 1, then the first number is a and the second number is $\frac{3}{4}$.

Thus, either $a = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$ or

$a = \frac{1}{2}\left(2 - \frac{3}{4}\right) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$.

Therefore, the possible first numbers in the chain are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.

We can double check that each of these gives a fourth number of 1:

$\frac{1}{8} \to \frac{1}{4} \to \frac{1}{2} \to 1$ $\quad$ $\frac{3}{8} \to \frac{3}{4} \to \frac{1}{2} \to 1$

$\frac{5}{8} \to \frac{3}{4} \to \frac{1}{2} \to 1$ $\quad$ $\frac{7}{8} \to \frac{1}{4} \to \frac{1}{2} \to 1$

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YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

Q1. If $f(x)=\cos^{-1}(x-x^2)+\sqrt{\left(1-\frac{1}{|x|}\right)}+\frac{1}{[x^2-1]}$, then find the domain of $f(x)$ (where [.] is the greatest integer function). **[Kr. Srivastava; Patna]**

Ans. Let $f_1(x) = \cos^{-1}(x - x^2)$

$f_1(x)$ is defined when $-1 \le x - x^2 \le 1$

$$\Rightarrow \quad \frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2} \qquad ...(i)$$

and let $f_2(x)=\sqrt{1-\frac{1}{|x|}}$

$f_2(x)$ is defined when $\left(1-\frac{1}{|x|}\right)\ge 0$

$$\Rightarrow \quad x \le -1 \text{ and } x \ge 1 \qquad ...(ii)$$

and let $f_3(x)=\frac{1}{[x^2-1]}$

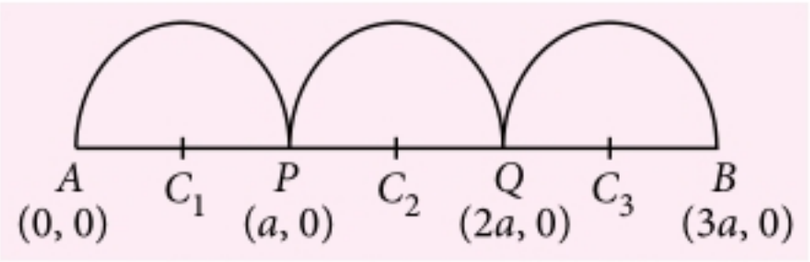
$f_3(x)$ is defined when $x^2 < 1, x^2 \ge 2$

$$\Rightarrow \quad x \in (-\infty, -\sqrt{2}] \cup (-1, 1) \cup [\sqrt{2}, \infty) \qquad ...(iii)$$

Then, $D_f = D_{f_1} \cap D_{f_2} \cap D_{f_3} = \left[\sqrt{2}, \frac{1+\sqrt{5}}{2}\right]$

Q2. Two points P and Q are taken on the line joining the points $A(0, 0)$ and $B(3a, 0)$ such that $AP = PQ = QB$. Circles are drawn on AP, PQ and QB as diameters. The locus of the points, the sum of the squares of the tangents from which to the three circles is equal to b^2, is **[Ragini; Bhopal]**

Ans. Since $AP = PQ = QB$. The coordinates of P are $(a, 0)$ and of Q are $(2a, 0)$.



The centre of the circles on AP, PQ and QB as diameters are respectively

$C_1\left(\frac{a}{2}, 0\right), C_2\left(\frac{3a}{2}, 0\right)$ and $C_3\left(\frac{5a}{2}, 0\right)$

and the radius of each one of them is $\left(\frac{a}{2}\right)$.

Hence, the equations of the circles with centre C_1, C_2 and C_3 are respectively

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}; \left(x-\frac{3a}{2}\right)^2+y^2=\frac{a^2}{4}$$

and $\left(x-\frac{5a}{2}\right)^2+y^2=\frac{a^2}{4}$.

So that, if $S(h, k)$ be any point on the locus, then

$$\left(h-\frac{a}{2}\right)^2+\left(h-\frac{3a}{2}\right)^2 + \left(h-\frac{5a}{2}\right)^2+3\left(k^2-\frac{a^2}{4}\right)=b^2$$

$$\Rightarrow \quad 3(h^2 + k^2) - 9ah + 8a^2 = b^2$$

Hence, the locus of $S(h, k)$ is

$$3(x^2 + y^2) - 9ax + 8a^2 - b^2 = 0$$

Q3. Find the value of $\lim_{n\to\infty} \{\log_{n-1}(n) \log_n (n + 1) \log_{n+1} (n + 2) ..... \log_{n^k-1} (n^k)\}$. **[Md. Aftab; Kolkata]**

Ans. $\lim_{n\to\infty}\left\{\frac{\log_e n}{\log_e(n-1)}\times\frac{\log_e(n+1)}{\log_e n}\times \frac{\log_e(n+2)}{\log_e(n+1)}\times ... \times\frac{\log_e n^k}{\log_e(n^k-1)}\right\}$

$$= \lim_{n\to\infty}\left\{\frac{\log_e n^k}{\log_e(n-1)}\right] = k \lim_{n\to\infty}\left\{\frac{\log_e n}{\log_e(n-1)}\right\}$$

Let $n=\frac{1}{h}$, then we have $k \lim_{h\to 0} \frac{\log_e\left(\frac{1}{h}\right)}{\log_e\left(\frac{1}{h}-1\right)}$

$$= k \lim_{h\to 0} \frac{-\log_e (h)}{\log_e (1-h)-\log_e (h)}$$

$$= k \lim_{h\to 0} \frac{1}{\left\{1-\frac{\log_e(1-h)}{\log_e(h)}\right\}} = k \cdot \frac{1}{(1-0)} = k$$

■■

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE(Main/ Advanced). This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE(Main/ Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. BC is latus rectum of a parabola $y^2 = 4ax$ and A is its vertex. The minimum length of projection of BC on a tangent drawn in portion BAC is

(a) $\sqrt{2}a$ (b) $2\sqrt{2}a$ (c) $2a$ (d) $3\sqrt{2}a$

2. Least value of the expression $\frac{1}{2bx-(x^2+b^2+\sin^2 x)}$, $x\in[-1, 0]$, $b \in [2, 3]$ is

(a) 1/4 (b) – 1/4

(c) $-\left(\frac{1}{8+\sin^2 1}\right)$ (d) 1/8

3. If in a right angled triangle ABC, $4\sin A \cos B - 1 = 0$ and $\tan A$ is real, then angles are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

4. If $|z| = 2$ and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{z-2}{z+2}$, then z_1, z_2, z_3 will be vertices of a/an

(a) equilateral triangle

(b) acute angled triangle

(c) right angled triangle (d) none of these

5. If $a^2 + b^2 - c^2 - 2ab = 0$, then the point(s) of concurrency of family of straight lines $ax + by + c = 0$ lie(s) on the line

(a) $y = x$ (b) $y = x + 1$

(c) $y = -x$ (d) $x + y = 1$

6. If $f''(x) > 0 \ \forall x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

(c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

7. A vector $\vec{r}$ is equally inclined with the vectors $\vec{a} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\vec{b} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ and $\vec{c} = \hat{k}$, then angle between $\vec{r}$ and $\vec{a}$ is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{2}$

8. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then one of the roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ is

(a) c (b) $d - c$ (c) $2c$ (d) $2d$

9. Let f be a differentiable function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y\ (\neq 0) \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x)$ is equal to

(a) $2f(x)$ (b) $\frac{f(x)}{x}$ (c) $2xf(x)$ (d) $\frac{2f(x)}{x}$

10. The number of solutions of the equation $\cos^{-1} x + \cos^{-1}\sqrt{1-x^2} = \pi$ is

(a) 1 (b) 2 (c) 0 (d) 3

SOLUTIONS

1. (b) : Let tangent at $P(at^2, 2at)$ make an angle θ with x axis, then $\tan\theta = \frac{1}{t}$

Projection of BC on tangent = $BC\sin\theta$

$$= \frac{4a}{\sqrt{1+t^2}} \geq 2a\sqrt{2} \quad (\text{as } -1 \leq t \leq 1).$$

2. (b) : Expression will be least if

$2bx - [x^2 + b^2 + \sin^2 x]$ is maximum

$\Rightarrow$ $x^2 + b^2 + \sin^2 x - 2bx$ is minimum

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

$\Rightarrow$ $(x-b)^2+\sin^2 x$ is minimum

Now $|x-b|$ and $|\sin x|$ are minimum if $x=0$, $b=2$.

So least value is $-\frac{1}{4}$.

3. (a) : Since $4\sin A\cos B=1$ so A and B cannot be $\frac{\pi}{2}$. [As if $B=\frac{\pi}{2}$, then $\cos B=0$ and if $A=\frac{\pi}{2}$, $\tan A$ is not defined]

$$\therefore \quad C=\frac{\pi}{2},\ B=\frac{\pi}{2}-A \Rightarrow 4\sin A\cos\left(\frac{\pi}{2}-A\right)=1$$

$$\Rightarrow \quad \sin^2 A=\frac{1}{4}\Rightarrow \sin A=\frac{1}{2}\Rightarrow A=\frac{\pi}{6}\Rightarrow B=\frac{\pi}{3}$$

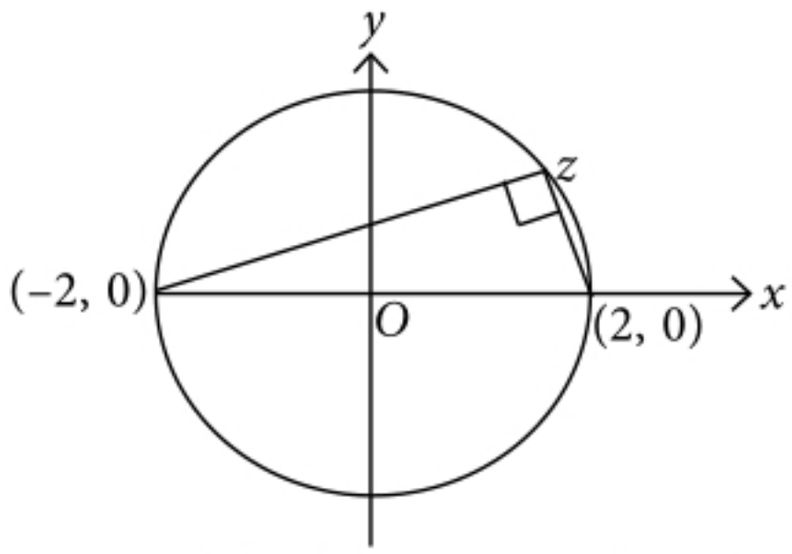
So angles are $\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ which are in A.P.

4. (c) : Since $|z|=2$

$\therefore$ z lies on a circle whose radius is 2.

$$\text{Clearly } \text{Arg}\left(\frac{z-2}{z+2}\right)=\pm\frac{\pi}{2}$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z_1-z_3}{z_2-z_3}\right)=\pm\frac{\pi}{2}$$



So z_1, z_2, z_3 will be the vertices of a right angled triangle.

5. (c) : $(a-b)^2-c^2=0$

$\Rightarrow$ $(a-b-c)(a-b+c)=0$

If $a-b=c \Rightarrow ax+by+a-b=0$

$\Rightarrow (x+1)a+b(y-1)=0 \Rightarrow x=-1,\ y=1$

If $-a+b=c \Rightarrow ax+by+b-a=0$

$\Rightarrow$ $(x-1)a+(y+1)b=0 \Rightarrow x=1,\ y=-1$

Equation of line passing through both points $(-1, 1)$ and $(1, -1)$ is $y=-x$.

6. (d) : $g'(x)=f'((\tan x-1)^2+3)(2\tan x-2)\sec^2 x$

Since $f''(x)>0 \Rightarrow f'(x)$ is increasing.

So, $f'((\tan x-1)^2+3)>f'(3)=0$

$$\forall\, x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

Also $(\tan x-1)>0 \ \forall x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

So $g(x)$ is increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$.

7. (c) : Since $\vec{a}\cdot\vec{b}=0=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a} \Rightarrow \vec{a},\vec{b},\vec{c}$ are mutually perpendicular.

Now $\vec{r}=t(\vec{a}+\vec{b}+\vec{c})$

$$=t((\cos\theta-\sin\theta)\hat{i}+(\cos\theta+\sin\theta)\hat{j}+\hat{k})$$

Let angle between $\vec{r}$ and $\vec{a}$ be α, then

$$\cos\alpha=\frac{\vec{r}\cdot\vec{a}}{|\vec{r}||\vec{a}|}=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

8. (b) : Let $f(x)=x^2+ax+b$, then

$x^2+(2c+a)x+c^2+ac+b=f(x+c)$

Thus roots of $f(x+c)=0$ will be $0, (d-c)$.

9. (d) : We have $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$

Replacing x and y both by 1, $f(1)=\frac{f(1)}{f(1)}=1$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\left\{\frac{\frac{f(x+h)}{f(x)}-1}{h}\right\}=f(x)\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)-1}{h}$$

$$=\frac{f(x)}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{h/x}$$

$$=\frac{f(x)}{x}f'(1)=\frac{2f(x)}{x} \quad (\because\ f'(1)=2)$$

10. (a) : $\cos^{-1}\sqrt{1-x^2}=\pi-\cos^{-1}x=\cos^{-1}(-x)$

$\Rightarrow$ $\sqrt{1-x^2}=-x \Rightarrow x<0$

Squaring, $1-x^2=x^2 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

Since $x<0$, $x=-\frac{1}{\sqrt{2}}$

PRACTICE PAPER 2015

JEE Main & Advanced

SECTION - I

SINGLE CORRECT CHOICE TYPE

This section contains 9 single correct choice type questions. Each question has 4 choices: (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. The complete set of values of real k such that equation $\tan^{-1}x - \cos^{-1}x = k$ has a real solution is

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left[-\frac{5\pi}{2}, \frac{\pi}{2}\right]$

(c) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (d) $\left[-\frac{5\pi}{4}, \frac{\pi}{4}\right]$

2. The plane containing the line $\frac{x-3}{2} = \frac{y-1}{4} = \frac{z-2}{5}$ is rotated through an angle $\pi/2$ about this line to contain the point $(4, 3, 7)$. The equation of plane before rotation is

(a) $4x + 8y + 8z = 4$ (b) $4x - 8y - 8z = 4$

(c) $4x + 8y - 8z = 4$ (d) $4x - 8y + 8z = 4$

3. The largest power of 2 that divides $(3+\sqrt{11})^{2n+1}$ is (where [.] is greatest integer function)

(a) n (b) $n + 1$

(c) $n - 1$ (d) none of these

4. Let S be the focus of $y^2 = 4x$ and a point P is moving on the curve such that its abscissa is increasing at the rate of 4 units/sec, then the rate of increase of projection of SP on $x + y = 1$ when P is at $(4, 4)$, is

(a) $\sqrt{3}$ (b) -1 (c) $-\sqrt{2}$ (d) $-\frac{3}{\sqrt{2}}$

5. If $S = (a-1)^2 + \left(\frac{b}{a} - 1\right)^2 + \left(\frac{c}{b} - 1\right)^2 + \left(\frac{4}{c} - 1\right)^2$, where $1 \le a \le b \le c \le 4$, then minimum value of S is

(a) $12 - 8\sqrt{2}$ (b) $12 - \sqrt{2}$

(c) $10 + 8\sqrt{2}$ (d) $16 - 2\sqrt{2}$

6. If A and B are two square matrices such that $A + B + AB = O$, then AB is equal to

(a) BA (b) A

(c) B (d) none of these

7. $\int_0^{\pi} [\cot x]\, dx$ is equal to (where [·] denotes greatest integer function)

(a) $-\pi/2$ (b) $\pi/2$ (c) $-\pi$ (d) π

8. A variable line L is drawn through origin to meet the lines $y - x - 10 = 0$ and $y - x - 20 = 0$ at the points A and B respectively. A point P is taken on L such that $\frac{1}{OP^2} = \frac{1}{OA^2} + \frac{1}{OB^2}$. Locus of P is

(a) $(y - x)^2 = 80$ (b) $(y - x)^2 = 100$

(c) $3x - 3y = 4$ (d) $3x - 3y = 10$

9. Lines $\frac{x}{a_1} + \frac{y}{b_1} = 1$ and $\frac{x}{a_2} + \frac{y}{b_2} = 1$ are not meeting co-ordinate axis on concyclic points. If $|a_1a_2| = |b_1b_2|$, then

(a) $a_1a_2 = b_1b_2$ (b) $a_1a_2 = \pm b_1b_2$

(c) $a_1a_2 = -b_1b_2$

(d) $|a_1a_2| = |b_1b_2|$ is not possible

SECTION-II

MULTIPLE CORRECT CHOICE TYPE

This section contains 4 multiple correct choice type questions. Each question has 4 choices: (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

10. If $(\sin^{-1}x + \sin^{-1}w)(\sin^{-1}y + \sin^{-1}z) = \pi^2$, then $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$; $N_1, N_2, N_3, N_4 \in N$

(a) has a maximum value 2
(b) has a minimum value –2
(c) is independent of N_1
(d) none of these

11. The function f is continuous and has the property $f(f(x)) = 1 - x$ for all $x \in [0, 1]$. Which of the following is/are true?

(a) $f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)=1$ (b) $\int_0^1 f(x)dx=\frac{1}{2}$

(c) $\lim_{x\to 1/2} f(x)=\frac{1}{2}$ (d) none of these

12. From a variable point on a fixed normal to a parabola, two other normals are drawn to parabola. Line joining their feet is
(a) parallel to a fixed line
(b) making acute angle with x-axis if slope of fixed normal is positive
(c) diameter to a parabola of fixed axis
(d) none of these

13. If a, b, c are the sides of a scalene triangle such that $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix}=0$, which of the following is/are not true?
(a) maximum value of the perimeter of the triangle is 3
(b) minimum value of $(1 + a)(1 + b)(1 + c)$ is 8
(c) maximum value of $(s - a)(s - b)(s - c)$ is 1/8.
(d) minimum value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is 3.

SECTION-III

INTEGER ANSWER TYPE

This section contains 7 questions.

The answer to each of the questions is a single digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers A, B, C and D (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following:

A	B	C	D
0	(0)	0	0
1	1	1	1
2	2	2	(2)
3	3	3	3
4	4	4	4
5	5	5	5
(6)	6	6	6
7	7	7	7
8	8	8	8
9	9	(9)	9

14. If $x + \sin y = 2010$ and $x + 2010\cos y = 2009$, $0 \le y \le \frac{\pi}{2}$, then find the value of $\frac{[x+y]}{1005}$ (where [.] denotes greatest integer function).

15. The tangent at any point P of a conic meets the y-axis in T such that OT is equal to twice the ordinate of P. If conic is passing through (1, 2), then find length of transverse axis of the conic.

16. Let $\vec{b}$ and $\vec{c}$ are non-collinear vectors. If $\vec{a}\times(\vec{b}\times\vec{c})+(\vec{a}\cdot\vec{b})\vec{b}=(4-2x-\sin y)\vec{b}+(x^2-1)\vec{c}$ and $(\vec{c}\cdot\vec{c})\vec{a}=\vec{c}$. Find $|x|$.

17. If $[x^2] + x - a = 0$ has a solution where $a \in N$ and $a \le 20$, then find total number of different possible values of a (where [.] denotes greatest integer function).

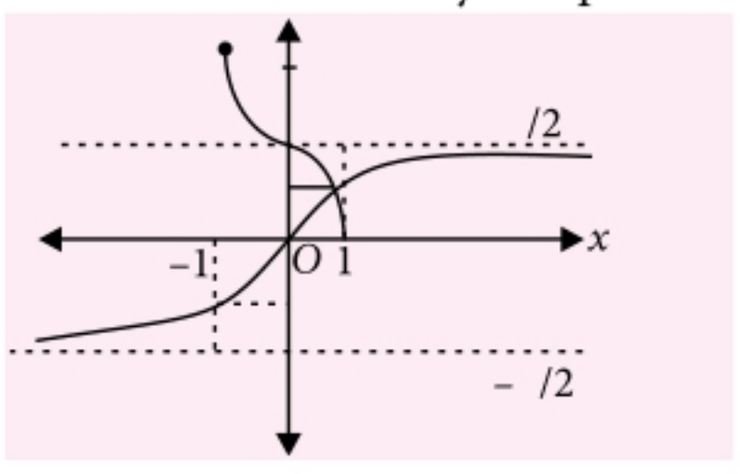
18. An odd function is symmetric about the vertical line $x = a$ $(a > 0)$ and if $\sum_{r=0}^{\infty}[f(1+4r)]^r=8$, then the numerical value of $8f(1)$ is

19. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals

20. If the area bounded by region $|y| + 1 \le 2e^{-|x|}$ is $A(1 - \log_e 2)$, then value of A is

SOLUTIONS

1. (d) : Clearly from graph if $k \in [-(\pi + \tan^{-1}1), \tan^{-1}1]$ both $\tan^{-1}x$ and $(\cos^{-1}x) + k$ will intersect and at exactly one point.



2. (c): Plane containing line will be given as
$(4x - 2y - 10) + \lambda(5y - 4z + 3) = 0$
$\Rightarrow$ $4x + (5\lambda - 2)y - 4\lambda z + 3\lambda - 10 = 0$, the plane passing through (4, 3, 7) will give λ
$\Rightarrow$ $16 + (5\lambda - 2)3 - 25\lambda - 10 = 0 \Rightarrow -10\lambda = 0 \Rightarrow$
$\lambda = 0$
i.e. $4x - 2y - 10 = 0$ ...(i)

If this is rotated through an angle $\frac{\pi}{2}$, the plane will be

$4x + (5\lambda' - 2)y - 4\lambda' z + (3\lambda' - 10) = 0$ ...(ii)

Angle between (i) and (ii) is $\frac{\pi}{2}$,

Also $16 + (5\lambda' - 2)(-2) = 0 \Rightarrow \lambda' = 2$;

$\therefore$ The equation of plane is $4x + 8y - 8z = 4$

3. (b) : $I + f = (\sqrt{11} + 3)^{2n+1}$

$f' = (\sqrt{11} - 3)^{2n+1}$

$I + f - f' = (\sqrt{11} + 3)^{2n+1} - (\sqrt{11} - 3)^{2n+1}$

$= (\sqrt{11} + 3)(20 + 6\sqrt{11})^n - (\sqrt{11} - 3)(20 - 6\sqrt{11})^n$

$= 2^n[\sqrt{11}(10 + 3\sqrt{11})^n - \sqrt{11}(10 - 3\sqrt{11})^n$

$+ 3(10 + 3\sqrt{11})^n + 3(10 - 3\sqrt{11})^n]$

$= 2^n[\sqrt{11} \times 2[{}^nC_1(10)^{n-1} \cdot 3\sqrt{11} + ...]$

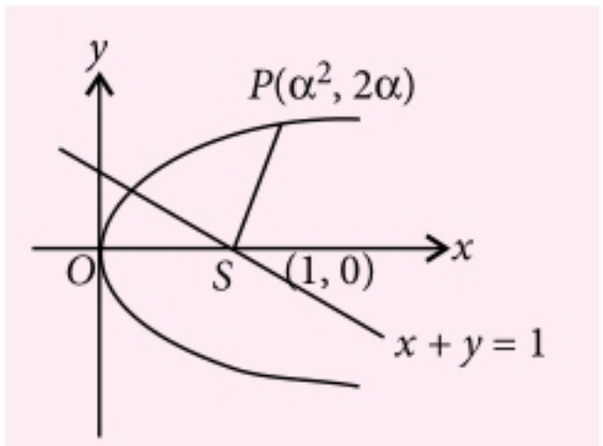
$+ 3 \times 2[{}^nC_0 10^n + ...]]$

$= 2^{n+1} \times$ integer, $f - f' = 0$; So, $I = 2^{n+1} \times$ integer

4. (c): $\overrightarrow{SP} = (\alpha^2 - 1)\hat{i} + 2\alpha\hat{j}$ unit vector along given line

$\hat{n} = \frac{\hat{j} - \hat{i}}{\sqrt{2}}$

$\therefore$ projection of SP on $x + y = 1$ is



$Z = \overrightarrow{SP}.\hat{n} = \frac{(1 - \alpha^2) + 2\alpha}{\sqrt{2}}$

$\Rightarrow \sqrt{2}Z = 1 - \alpha^2 + 2\alpha \Rightarrow \sqrt{2}\frac{dZ}{dt} = -2\alpha\frac{d\alpha}{dt} + 2\frac{d\alpha}{dt}$

Given $\frac{dx}{dt} = 4$ but $x = \alpha^2 \Rightarrow \frac{dx}{dt} = 2\alpha\frac{d\alpha}{dt} = 4$

when $P(4, 4)$ then $\alpha = 2 \Rightarrow \frac{d\alpha}{dt} = 1$

$\therefore \sqrt{2}\frac{dZ}{dt} = -4 + 2 = -2 \Rightarrow \frac{dZ}{dt} = -\sqrt{2}$

5. (a) : Given,

$S = (a-1)^2 + \left(\frac{b}{a} - 1\right)^2 + \left(\frac{c}{b} - 1\right)^2 + \left(\frac{4}{c} - 1\right)^2$

$1 \le a \le b \le c \le 4$

$\therefore \quad a.\frac{b}{a}.\frac{4}{c}.\frac{c}{b} = 4$

$\Rightarrow \quad a = x, \frac{b}{a} = y, \frac{c}{b} = z, \frac{4}{c} = u, \ xyzu = 4$

Consider a case $= (x-1)^2 + (y-1)^2$ ($\because \quad xy = \lambda$)

$f(x) = (x-1)^2 + \left(\frac{\lambda}{x} - 1\right)^2$

$f'(x) = 2(x-1) - 2\left(\frac{\lambda}{x} - 1\right).\frac{\lambda}{x^2} = 0$

$\Rightarrow \quad (x^2 - \lambda)(x^2 - x + \lambda) = 0$

Thus, real roots are $-\sqrt{\lambda}$ and $\sqrt{\lambda}$ where $\sqrt{\lambda}$ is point of minima so gives least value.

When $x = y = \sqrt{\lambda}$

Thus for given question $x = y = z = u = \sqrt{2}$

Thus, value is $(12 - 8\sqrt{2})$

6. (a) : Given $AB + A + B = O$

$\Rightarrow \quad A + B + AB + I = I \quad (\because \quad I$ is an identity matrix)

$\Rightarrow \quad (A + I)(B + I) = I$

$\Rightarrow \quad A + I$ and $B + I$ is inverse to each other

then $(B + I)(A + I) = (A + I)(B + I)$

$\Rightarrow \quad AB = BA$

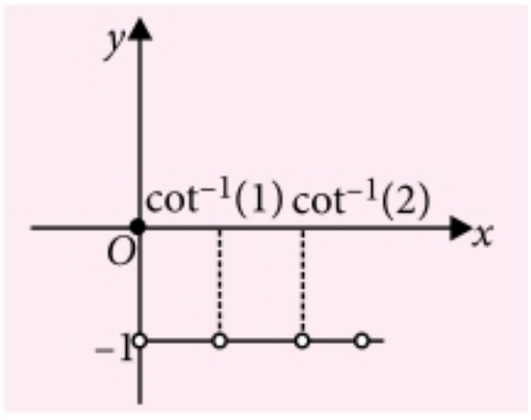
7. (a) : $I = \int_0^{\pi} [\cot x]\, dx$ ...(i)

$I = \int_0^{\pi} [\cot(\pi - x)]\, dx = \int_0^{\pi} [-\cot x]\, dx$ ...(ii)

(i) + (ii) $\Rightarrow \quad 2I = \int_0^{\pi} ([\cot x] + [-\cot x])\, dx$

Now $[\cot x] + [-\cot x] = \begin{cases} 0 & , \text{if } \cot x \in Z \\ -1 & , \text{if } \cot x \notin Z \end{cases}$

So graph is as



$\Rightarrow \quad 2I = -\pi \Rightarrow I = -\frac{\pi}{2}$

8. (a) : Let equation of line is $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r$

$\Rightarrow$ ($OA\cos\theta$, $OA\sin\theta$), ($OB\cos\theta$, $OB\sin\theta$)will satisfy

$y - x - 10 = 0$ and $y - x - 20 = 0$ respectively.
If $P(r\cos\theta, r\sin\theta)$, then

$$\frac{1}{r^2} = \left(\frac{\sin\theta - \cos\theta}{10}\right)^2 + \left(\frac{\sin\theta - \cos\theta}{20}\right)^2$$

$\Rightarrow$ $(r\sin\theta - r\cos\theta)^2 = 80$

$\therefore$ Locus of P is $(y - x)^2 = 80$

9. (c): If points $(a_1, 0)$ and $(a_2, 0)$ are on same side of x-axis with respect to origin, then $(0, b_1)$, $(0, b_2)$ must be on opposite sides.

10. (a, b) : x, y, z, w are all 1 or all -1.
So the determinant has all elements 1 or -1,
hence possible values of determinant are $-2, 0, 2$.

11. (a, b, c) : $f(x) = f^{-1}(1 - x)$
Also, $f(x) = 1 - f^{-1}(x)$

$$\Rightarrow \quad f(x) + f(1-x) = 1$$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = 1 \Rightarrow f\left(\frac{1}{2}\right) = \frac{1}{2}$$

12. (a, b, c) : Let parabola be $y^2 = 4ax$ and any variable point of N is (h, k)
Let feet of other two normals are P and Q
Equation of normal to parabola is

$$y = mx - 2am - am^3$$

$\Rightarrow$ m_1, m_2, m_3 are slopes of these tangents and they are roots of equation

$\Rightarrow$ $am^3 + (2a - h)m + k = 0 \Rightarrow m_1 + m_2 + m_3 = 0$

$$\text{Also slope of } PQ = \frac{-2a(m_2 - m_3)}{a(m_2 - m_3)(m_2 + m_3)}$$

$$= -\frac{2}{m_2 + m_3} = \frac{2}{m_1} \text{ (Fixed)}$$

13. (a, b, c, d) : $abc = 1$

$$\Rightarrow \quad \frac{a+b+c}{3} \geq 1, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$$

Also $(1 + a)(1 + b)(1 + c) \geq 2^3$ and

$$(b + c - a)(c + a - b)(a + b - c) \leq 1$$

But a, b, c cannot be equal.

14. (2) : $x + \sin y = 2010$ ...(i)

$x + 2010\cos y = 2009$ ...(ii)

(i) – (ii) $\Rightarrow$ $\sin y = 1 + 2010\cos y$

It can be possible if $y = \frac{\pi}{2}$

$$\Rightarrow \quad x = 2009 \Rightarrow \frac{[x+y]}{1005} = \frac{\left[2009 + \frac{\pi}{2}\right]}{1005} = 2$$

15. (4) : Equation of tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad T\left(0, -x\frac{dy}{dx} + y\right) \Rightarrow \quad -x\frac{dy}{dx} + y = 2y$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0 \Rightarrow \quad xy = c^2$$

$\Rightarrow$ Conic is $xy = 2$

16. (1) : $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b}$

$$= (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$$

$\Rightarrow$ $\vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} = 4 - 2x - \sin y$,
and $-\vec{a}\cdot\vec{b} = x^2 - 1$. Also, $\vec{a}\cdot\vec{c} = 1$

$\Rightarrow$ $1 + 1 - x^2 = 4 - 2x - \sin y$

$\Rightarrow$ $\sin y = x^2 - 2x + 2 = (x - 1)^2 + 1$

$\Rightarrow$ $(x - 1)^2 + 1 = 1$ $\quad (\because \sin y \leq 1)$

$\Rightarrow$ $x = 1$

17. (4) : $x = a - [x^2] \Rightarrow x \in Z$

$\Rightarrow$ $a = x(x + 1)$

So a can be 2, 6, 12, 20

18. (7) : $f(2a - x) = f(x)$

$\Rightarrow$ $f(2a + x) = -f(x)$

$\because$ f is odd $\Rightarrow$ $f(x + 4a) = f(x)$

$\Rightarrow$ f is periodic with period $4a$

$\Rightarrow$ $f(1 + 4r) = f(1)$

$$\text{Now } \sum_{r=0}^{\infty} [f(1)]^r = 8 \Rightarrow \frac{1}{1 - f(1)} = 8$$

$\Rightarrow$ $f(1) = 7/8 \quad \Rightarrow \quad 8f(1) = 7$

19. (1) : $f(x) = x + \cos x + 2$, $f(0) = 3$

$\Rightarrow$ $g(3) = 0$

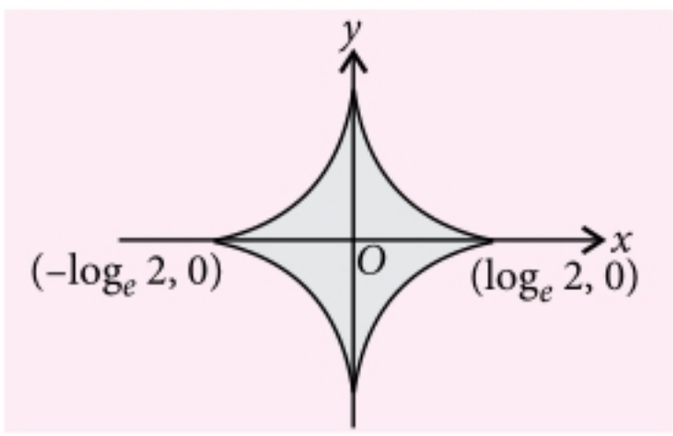
$g(f(x)) = x \Rightarrow g'(f(x))\cdot f'(x) = 1$.

Putting $x = 0$, $g'(3)\cdot f'(0) = 1$

Now, $f'(x) = 1 - \sin x \Rightarrow f'(0) = 1 \Rightarrow g'(3) = 1$.

20. (4) : Area is

$$4\int_0^{\log_e 2} (2e^{-x} - 1)\,dx = 4(1 - \log_e 2)$$



15 Challenging Problems For Entrance Exams

SECTION-1

ONLY ONE CORRECT OPTION

1. If $f(x+1)=\frac{f(x)-1}{f(x)+1}\ \forall\ x\in N$ with $f(1)=2$, then $\int_1^5 \sum_{r=1}^{1728} \frac{1}{f(r)}dx=$

(a) 2014 (b) 2015
(c) 2016 (d) none of these

2. If $t = ax^2 + 2bx + c$, $y = at^2 + 2bt + c$, then $\int_{-1}^{1}\left(\frac{d^4y}{dx^4}\right)dx=$

(a) $48a^3$ (b) $24a^2$
(c) $24a^3$ (d) none of these

3. $\int_{-1}^{1}\left[1008+\frac{[x]}{x^{2014}+1}\right]dx=$
(where [·] denotes greatest integer function)

(a) 0 (b) 1008
(c) 2016 (d) 2015

4. If $a = 5$, $b = 6$, $A = 60°$ in ΔABC, then

(a) $\tan B=\frac{3\sqrt{3}}{2}$ (b) $B=\sin^{-1}\frac{3\sqrt{3}}{5}$
(c) $\cos B<0$ (d) none of these

5. If α, β, γ be the angles made by a line with axes so that

$$2\left(\frac{\tan^2\alpha}{1+\tan^2\alpha}+\frac{\tan^2\beta}{1+\tan^2\beta}+\frac{\tan^2\gamma}{1+\tan^2\gamma}\right)=3\sec^2\frac{\theta}{2},$$

then $\theta =$

(a) $\pi/12$ (b) $\pi/10$ (c) $\pi/6$ (d) $\pi/3$

6. $\int_1^9\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\log_3 t}\right)dt+\int_1^2\frac{3^{x^2-[x]}}{3^{\{x\}}}dx=$

(a) 80 (b) 8
(c) 17 (d) can't be determined

SECTION-II

ONE OR MORE THAN ONE OPTION CORRECT

7. If $\alpha=\int_0^1 e^{x^2}dx-e, \beta=\int_0^1 e^{x^2}dx-1$, then

(a) $\alpha\beta>0$ (b) $\frac{\alpha}{\beta}<0$ (c) $\frac{\beta}{\alpha}<0$ (d) $\alpha\beta<0$

8. If $\alpha=\int_1^3[x]^{\{x\}}dx$ and $\beta=\int_1^3\{x\}^{[x]}dx$, then

(a) α is irrational but β is rational.
(b) α is rational but β is irrational.
(c) $5\alpha - 6\beta$ is irrational.
(d) $\frac{\alpha-1}{\alpha-\beta}$ is irrational.

9. $\int_0^{\pi/2} e^{\sqrt{\sin x}}\cdot\sin 2x dx$ belongs to

(a) $(2, e)$ (b) $(e-1, \pi-1)$
(c) (e, π) (d) $[1, e]$

10. If $\frac{2014\alpha+2015}{1007\alpha}\int_0^{1007}\left(\left\{\sin\frac{\pi x}{2014}\right\}+\left\{\sin\left(\frac{-\pi x}{2014}\right)\right\}\right)dx=2015$

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(where $\{x\}$ denotes fractional part function}, then $\alpha =$

(a) 1007 (b) 2014

(c) 2015 (d) none of these

11. The digit in unit place of $2^{(9^{100}-1)}$ is divisible by

(a) 2 (b) 4 (c) 3 (d) 6

12. If the cubic equation $x^3 + k = 3x$ has all integral roots, then

(a) $|k|$ is prime.

(b) $[k]$ satisfy $x^2 - 4 = 0$.

(c) $|[k]|$ can be eccentricity of an ellipse.

(d) k^2 is the area of the triangle formed by $x - 2y = 4$ with axes.

13. If z_1, z_2, z_3 be the vertices of an equilateral triangle ABC and satisfy $z^3 + 3\alpha z + \beta = 3\gamma z^2$, then equation of tangent to $x = y^2$ at $P(\alpha^2, \gamma^2)$ will be

(a) $x - 2\alpha y + \alpha^2 = 0$ (b) $x = \gamma^2(2y - \alpha)$

(c) $x + \gamma^4 = 2\alpha y$ (d) $x - 2\alpha y + \beta = 0$

14. If $f(x)$ be a non-negative function so that $|41 + f(x) - 4x^2| - 5 = |f(x)| + 4|9 - x^2|$ is true for $x \in [\alpha, \beta]$, then $\int_{\alpha}^{\beta} (ax^{\beta} + bx^{\beta-2} + cx^{\alpha+\beta})dx$ will not depend on

(a) a (b) b (c) c (d) a, b, c

15. If $f''(x)$ is continuous $\forall\ x \in R$ and $f(0) = 1 = f'(1)$ such that $\int_0^{f(0)} zf''(z)dz$ vanishes, then

(a) $f(1) = f(0) + f'(1)$ (b) $f'(1) = \sum_{r=1}^{\infty} \frac{1}{\{f(1)\}^r}$

(c) $f(0), f(1), f'(1)$ are in A.P.

(d) $\sin^{-1}(\sin(f(1))$ is defined

SOLUTIONS

1. (c): $f(1) = 2$; $f(2) = \frac{2-1}{2+1} = \frac{1}{3}$;

$$f(3) = \frac{\frac{1}{3} - 1}{\frac{1}{3} + 1} = -\frac{1}{2};\quad f(4) = \frac{-\frac{1}{2} - 1}{-\frac{1}{2} + 1} = -3;$$

$$f(5) = \frac{-3-1}{-3+1} = 2;\quad f(6) = \frac{2-1}{2+1} = \frac{1}{3} \text{ etc.}$$

$\therefore\ f(1) = f(5) = f(9) = \ldots = 2$

$f(2) = f(6) = \ldots = 1/3$; $f(3) = f(7) = \ldots = -1/2$

$f(4) = f(8) = \ldots = -3$

The above result shows that $f(x)$ is periodic with period 4.

Now, $\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} = \frac{1}{2} + 3 - 2 - \frac{1}{3} = \frac{7}{6}$

$$\therefore \int_1^5 \sum_{r=1}^{1728} \frac{1}{f(r)} dx$$

$$= \int_1^5 \left[\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(1)} + \frac{1}{f(2)} + \ldots \text{to 1728 terms} \right] dx$$

$$= \int_1^5 432 \left\{ \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} \right\} dx$$

$$\left[\because\ \frac{1728}{4} = 432 \right]$$

$$= 432 \int_1^5 \frac{7}{6} dx = 504 \left[x \right]_1^5 = 2016$$

2. (a) : $\frac{dt}{dx} = 2ax + 2b$; $\frac{dy}{dt} = 2at + 2b$

$$\therefore\ \frac{dy}{dx} = 4(ax + b)(at + b)$$

$$\Rightarrow\ \frac{d^2y}{dx^2} = 4\left[a(at + b) + (ax + b) \cdot a \cdot \frac{dt}{dx} \right]$$

$$= 4a\left[(at + b) + (ax + b) \cdot 2(ax + b) \right]$$

$$= 4a[(at + b) + 2(ax + b)^2]$$

$$\Rightarrow\ \frac{d^3y}{dx^3} = 4a\left[a \cdot \frac{dt}{dx} + 2 \cdot 2(ax + b)a \right]$$

$$= 4a^2[2(ax + b) + 4(ax + b)] = 24a^2(ax + b)$$

Now, $\int_{-1}^{1} \left(\frac{d^4y}{dx^4} \right) dx = \left[\frac{d^3y}{dx^3} \right]_{-1}^{1}$

$$= 24a^2 \left[ax + b \right]_{-1}^{1} = 24a^2 \cdot 2a = 48a^3$$

3. (d) : If $-1 \le x < 0$, $[x] = -1$ and $1 < x^{2014} + 1 \le 2$

$$\Rightarrow\ \frac{1}{2} \le \frac{1}{x^{2014} + 1} < 1 \Rightarrow\ -\frac{1}{2} \ge \frac{-1}{x^{2014} + 1} > -1$$

$$\Rightarrow\ -\frac{1}{2} \ge \frac{[x]}{x^{2014} + 1} > -1\ \ [\because\ [x] = -1]$$

$$\Rightarrow \quad 1007\frac{1}{2} \geq 1008 + \frac{[x]}{x^{2014}+1} > 1007$$

$$\therefore \quad \left[1008 + \frac{[x]}{x^{2014}+1}\right] = 1007$$

If $0 \leq x < 1$, $[x] = 0$ and so $\dfrac{[x]}{x^{2014}+1} = 0$

$$\Rightarrow \quad 1008 + \frac{[x]}{x^{2014}+1} = 1008$$

$$\therefore \quad \left[1008 + \frac{[x]}{x^{2014}+1}\right] = 1008$$

$$\therefore \quad I = \int_{-1}^{0} 1007\,dx + \int_{0}^{1} 1008\,dx = 2015$$

4. (d) : Using $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, we have

$$\frac{5}{(\sqrt{3}/2)} = \frac{6}{\sin B} \quad \Rightarrow \quad \sin B = \frac{3\sqrt{3}}{5} \text{(Absurd)}$$

5. (d) : We know that,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \Rightarrow \quad \Sigma\cos^2\alpha = 1 \quad \ldots$ (i)

Given that

$$2\left(\frac{\tan^2\alpha}{1+\tan^2\alpha} + \frac{\tan^2\beta}{1+\tan^2\beta} + \frac{\tan^2\gamma}{1+\tan^2\gamma}\right) = 3\sec^2\frac{\theta}{2}$$

$$\Rightarrow \quad 2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 3\sec^2\frac{\theta}{2}$$

$$\Rightarrow \quad 2(1-\cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma) = 3\sec^2\frac{\theta}{2}$$

$$\Rightarrow \quad 2(3 - \Sigma\cos^2\alpha) = 3\sec^2\frac{\theta}{2} \quad \Rightarrow \quad 4\cos^2\frac{\theta}{2} = 3$$

[From (i)]

$$\Rightarrow \quad 2(1+\cos\theta) = 3 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

6. (c): Let $y = f(x) = \dfrac{3^{x^2-[x]}}{3^{\{x\}}}$

$$\therefore \quad f(x) = 3^{x^2-([x]+\{x\})} = 3^{x^2-x} = y \qquad \ldots \text{(i)}$$

$$\therefore \quad x^2 - x - \log_3 y = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+4\log_3 y}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \log_3 y}$$

$$\therefore \quad x = \frac{1}{2} + \sqrt{\frac{1}{4} + \log_3 y} \quad \text{as } y \geq 1 \text{ if } x \geq 1 \text{ [from (i)]}$$

$$\therefore \quad f^{-1}(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + \log_3 x}$$

Also, $f(1) = 1$ and $f(2) = 9$

$$\therefore \quad I = \int_{1}^{9}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \log_3 t}\right)dt + \int_{1}^{2} 3^{x^2-x}\,dx$$

$$= \int_{1}^{9} f^{-1}(t)\,dt + \int_{1}^{2} 3^{t^2-t}\cdot dt \;= \int_{f(1)}^{f(2)} f^{-1}(t)\,dt + \int_{1}^{2} f(t)\,dt$$

$= 2f(2) - 1\,f(1) = 2(9) - 1(1) = 17$

Note : $\displaystyle\int_{a}^{b} f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a)$

7. (b, c, d) : In $(0, 1)$, $x^2 < 1 \quad \Rightarrow \quad e^{x^2} < e^1$

$$\Rightarrow \quad \int_{0}^{1} e^{x^2}\,dx < \int_{0}^{1} e\,dx = \int_{0}^{1} e^{x^2}\,dx - e < 0 \quad \Rightarrow \quad \alpha < 0$$

Also, $x^2 > 0$ in $(0, 1)$

$$\Rightarrow \quad e^{x^2} > e^0 \quad \Rightarrow \quad \int_{0}^{1} e^{x^2}\,dx > \int_{0}^{1} e^0\,dx$$

$$\Rightarrow \quad \int_{0}^{1} e^{x^2}\,dx - 1 > 0 \quad \Rightarrow \quad \beta > 0$$

$$\therefore \quad \alpha\beta < 0, \quad \frac{\alpha}{\beta} < 0, \quad \frac{\beta}{\alpha} < 0$$

8. (a, c, d) : $\alpha = \displaystyle\int_{1}^{2} 1^{x-1}\,dx + \int_{2}^{3} 2^{x-2}\,dx = 1 + \frac{1}{\log_e 2}$

which is irrational.

And, $\beta = \displaystyle\int_{1}^{2} (x-1)^1\,dx + \int_{2}^{3} (x-2)^2\,dx$

$$= \left[\frac{x^2}{2} - x\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 = \frac{1}{2} + \left(\frac{1}{3} - 0\right) = \frac{5}{6}$$

which is rational.

$$\therefore \quad 5\alpha - 6\beta = \frac{5}{\log_e 2}, \text{ which is irrational.}$$

Also, $\dfrac{\alpha-1}{\alpha-\beta}$ is also irrational.

9. (a, d) : $I = \displaystyle\int_{0}^{1} e^z \cdot 2z^2 \cdot 2z\,dz$

Put $\sin x = z^2$, $\cos x\,dx = 2z\,dz$

x	0	$\pi/2$
z	0	1

$$= 4\left[(z^3 - 3z^2 + 6z - 6)e^z\right]_0^1$$

$= 4[-2e + 6] = 4(6 - 5.436564)$ (app.)

$(\because\ e = 2.718282$ (app.)$)$

$= 2.253744$ (app.) $\therefore$ (a) and (d) are correct.

10. (c): $\because$ In $(0, 1007)$, $\sin\dfrac{\pi x}{2014} \in (0, 1)$

and so, $\sin\dfrac{\pi x}{2014}$ is not an integer.

Since, $\{x\} + \{-x\} = 1$, if $x \notin I$

$$\therefore \left\{\sin\frac{\pi x}{2014}\right\}+\left\{\sin\left(\frac{-\pi x}{2014}\right)\right\}=\left\{\sin\frac{\pi x}{2014}\right\}+\left\{-\sin\frac{\pi x}{2014}\right\}=1$$

$\therefore$ We have,

$$\frac{2014\alpha+2015}{1007\alpha}\int_0^{1007}\left(\left\{\sin\frac{\pi x}{2014}\right\}+\left\{\sin\left(\frac{-\pi x}{2014}\right)\right\}\right)dx = 2015$$

gives $\dfrac{2014\alpha+2015}{1007\alpha}\displaystyle\int_0^{1007} 1\,dx = 2015$

$$\Rightarrow \frac{2014\alpha+2015}{\alpha}=2015 \Rightarrow \alpha=2015$$

11. (a, c, d) : $9^{100} - 1 = (1 + 8)^{100} - 1$

$= {}^{100}C_0 \cdot 8 + {}^{100}C_2 \cdot 8^2 + ... = 8n$ (say)

Also, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$ and any integral power of 16 always have 6 at its unit place.

$\therefore\ 2^{(9^{100}-1)} = 2^{8n} = 16^{2n}$ will have 6 at its units place.

12. (a, b, d) : Let α, β, γ be the integral roots of $x^3 + k = 3x$

$\therefore\ \Sigma\alpha = 0$ ... (1), $\Sigma\alpha\beta = -3$ ... (2)

$\alpha\beta\gamma = -k$ ... (3)

$\because\ \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 6 \Rightarrow |\alpha|, |\beta|, |\gamma| \in \{0, 1, 2\}$

$\therefore\ \Sigma\alpha = 0$ and $\Sigma\alpha^2 = 6 \Rightarrow \alpha = 1, \beta = 1, \gamma = -2$

or $\alpha = -1, \beta = -1, \gamma = 2$

Only for these values, (2) is satisfied.

$\therefore$ (3) gives $k = \pm 2 \Rightarrow |k| = 2$ and $[k] = \pm 2$

$\Rightarrow |k| = 2$, which is prime; $[k]$ satisfy $x^2 - 4 = 0$

$\because$ $|[k]| = 2$, which can't be eccentricity of any ellipse (as $e < 1$).

Also, $x - 2y = 4$ *i.e.* $\dfrac{x}{4}+\dfrac{y}{-2}=1$ forms an area $= 4 = k^2$ with axes.

13. (a, b, c) : $z^3 - 3\gamma z^2 + 3\alpha z + \beta = 0$

Its roots are z_1, z_2 and z_3

$\therefore\ z_1 + z_2 + z_3 = 3\gamma$; $z_1z_2 + z_2z_3 + z_3z_1 = 3\alpha$

and $z_1z_2z_3 = -\beta$

For equilateral ΔABC, we know that $\Sigma z_1^2 = \Sigma z_1z_2$

$\Rightarrow (\Sigma z_1)^2 - 2\Sigma z_1z_2 = \Sigma z_1z_2$

$\Rightarrow (3\gamma)^2 = 3(3\alpha) \Rightarrow \alpha = \gamma^2$

From $x = y^2$, we have, $1 = 2y\cdot\dfrac{dy}{dx}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\text{at } P(\alpha^2,\ \gamma^2)} = \frac{1}{2\gamma^2} = \frac{1}{2\alpha}$$

$\therefore$ Equation of tangent to $x = y^2$ at $P(\alpha^2, \gamma^2)$ will be

$$y-\gamma^2=\frac{1}{2\alpha}(x-\alpha^2)$$

$\Rightarrow\ 2\alpha y - 2\alpha^2 = x - \alpha^2$ [Since $\gamma^2 = \alpha$]

$\Rightarrow\ x - 2\alpha y + \alpha^2 = 0$ or $x = \gamma^2(2y - \alpha)$

or $x + \gamma^4 = 2\alpha y$

14. (a, b): $\because\ 41 + f(x) - 4x^2 = f(x) + (36 - 4x^2) + 5$

$\therefore\ |41 + f(x) - 4x^2| = |f(x) + (36 - 4x^2) + 5|$

$\le |f(x)| + |36 - 4x^2| + |5|$

$\Rightarrow\ |41 + f(x) - 4x^2| \le |f(x)| + 4|9 - x^2| + 5$

Equality will hold only if $f(x)\cdot(9 - x^2) \ge 0$ [$\because\ 5 > 0$]

i.e. $|41 + f(x) - 4x^2| - 5 = |f(x)| + 4|9 - x^2|$ only if $f(x)\,(9 - x^2) \ge 0$

$\Rightarrow\ 9 - x^2 \ge 0$ [Since $f(x) \nless 0$]

$\Rightarrow\ x^2 - 9 \le 0 \Rightarrow -3 \le x \le 3$

$\Rightarrow\ x \in [-3, 3] \Rightarrow \alpha = -3, \beta = 3$

Now, $\displaystyle\int_\alpha^\beta (ax^\beta + bx^{\beta-2} + cx^{\alpha+\beta})dx$

$$= \int_{-3}^{3}(ax^3+bx+c)dx = \int_{-3}^{3} c\,dx$$

[$\because\ ax^3 + bx$ is an odd function]

$= 6c$, which does not depend on a and b.

15. (a, b, d) : Since $f(0) = 1 = f'(1)$ ... (1)

And, $\displaystyle\int_0^{f(0)} zf''(z)dz$ vanishes

$$\therefore \int_0^1 zf''(z)dz = 0 \Rightarrow \left[zf'(z)\right]_0^1 - \int_0^1 1\cdot f'(z)dz = 0$$

$$\Rightarrow f'(1)-\left[f(z)\right]_0^1=0 \Rightarrow f'(1)-f(1)+f(0)=0$$

$\Rightarrow\ 1 - f(1) + 1 = 0$ [using (1)]

$\Rightarrow\ f(1) = 2 \quad \therefore\ f(1) = f(0) + f'(1)$

$$\sum_{r=1}^{\infty}\frac{1}{\{f(1)\}^r}=\frac{1}{f(1)}+\frac{1}{\{f(1)\}^2}+\frac{1}{\{f(1)\}^3}+\ldots \text{ to } \infty$$

$$=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+.....=\frac{1/2}{1-\frac{1}{2}}=1=f'(1)$$

$$\therefore\ f'(1)=\sum_{r=1}^{\infty}\frac{1}{\{f(1)\}^r}$$

$f(0), f(1), f'(1)$ are not in A.P.

$$\sin^{-1}(\sin f(1)) = \sin^{-1}(\sin 2) = \pi - 2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\therefore\ \sin^{-1}(\sin f(1))$ is defined.

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

PAPER-I

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct.

1. If $f(x)$ and $g(x)$ are functions such that $f(x+y) = f(x)\cdot g(y) + g(x)\cdot f(y)$, then

$$\begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha+\theta) \\ f(\beta) & g(\beta) & f(\beta+\theta) \\ f(\gamma) & g(\gamma) & f(\gamma+\theta) \end{vmatrix}$$ is independent of

(a) α (b) β (c) γ (d) θ

2. Sum of the roots of the equation $(x+1) = 2\log_2(2^x+3) - 2\log_4(1980 - 2^{-x})$ is greater than

(a) 2 (b) 3 (c) 4 (d) 5

3. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)\cdot P(B)$ then

(a) $P(B/A) = \dfrac{P(B)}{P(A)}$

(b) $P(A^c \cup B^c) = P(A^c) + P(B^c)$

(c) $P((A \cup B)^c) = P(A^c)P(B^c)$

(d) $P(A/B) = P(A)$

4. The origin and roots of the equation $z^2 + pz + q = 0$ form an equilateral triangle if

(a) $p^2 = q$ (b) $p^2 = 3q$

(c) $q^2 = 3p$ (d) $q^2 = p$

5. Let $\{\Delta_1, \Delta_2, \Delta_3, \ldots\ldots, \Delta_k\}$ be the set of third-order determinants that can be made with the distinct non zero real numbers $a_1, a_2, a_3, \ldots\ldots, a_9$. Then

(a) $k = 9!$ (b) $\sum_{i=1}^{k} \Delta_i = 0$

(c) at least one $\Delta_i = 0$ (d) None of these

6. The quadratic equation $x^2 - 2x - \lambda = 0$, $\lambda \neq 0$,

(a) cannot have a real roots if $\lambda < -1$.

(b) can have a rational root if λ is a perfect square of an integer.

(c) cannot have an integral root if $n^2 - 1 < \lambda < n^2 + 2n$, where $n = 0, 1, 2, 3, \ldots\ldots$

(d) none of these

7. Given the set of four real numbers. First three numbers are in G.P. and last three numbers are in A.P. The common difference of A.P. is 6. First and fourth terms are same. According to the given data,

(a) fourth term = 4 (b) fourth term = 8

(c) sum of 4 nos. = 14 (d) sum of 4 nos. = 16

8. If $x + y + z = 5$ and $xy + yz + zx = 3$; $z, y, z \in R$ then for x

(a) largest value $= \dfrac{13}{3}$ (b) largest value $= \dfrac{10}{3}$

(c) least value = -1 (d) least value = 5

9. In the expansion of $\left(\sqrt[3]{4} + \dfrac{1}{\sqrt[4]{6}}\right)^{20}$,

(a) the number of irrational terms = 19

(b) middle term is irrational

(c) the number of irrational terms = 15

(d) 9^{th} term is rational

10. Let $S_n = \dfrac{3}{2}\cdot\dfrac{1}{1^2} \cdot + \dfrac{5}{2}\cdot\dfrac{1+2}{1^2+2^2} + \dfrac{7}{2}\cdot\dfrac{1+2+3}{1^2+2^2+3^2} + \ldots\ldots$

upto n terms, $n \in N$, then S_n is not greater than

(a) n (b) $2n$ (c) $3n$ (d) 1

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive).

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11. Find the number of roots of the equation $z^{10} = 1$ satisfying $|\arg z| < (\pi/2)$.

12. Find the number of quadratic equations which remain unchanged by squaring their roots.

13. A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If A is the event, 'both head and tail have appeared', and B be the event, 'at most one tail is observed', then $5[P(B/A)]$ is

14. The equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $3a + b + 3$ is

15. Find the number of solution of the equation, $|x - x^2 - 1| = |2x - 4 - x^2|$.

16. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio $6 : 33 : 110$, then $\frac{n}{2}$ is

17. The sum of the infinite terms of the sequence $\frac{5}{3^2 \cdot 7^2} + \frac{9}{7^2 \cdot 11^2} + \frac{13}{11^2 \cdot 15^2} + \ldots\ldots\ldots$ is $\frac{1}{8k}$ if $k =$

18. The number of positive integral solutions of $\frac{x^2(3x-4)^3(x-2)^4}{(x-5)^5(2x-7)^6} \leq 0$ is

19. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{20}x^{20}$, then $\frac{a_2}{30} =$

20. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$ and $\text{Arg}(z) - \text{Arg}(\omega) = \pi/2$, then $|\bar{z}\omega|$ is equal to

PAPER-II

SECTION-I

SINGLE CORRECT OPTION

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. On a railway there are 10 stations. The number of types of tickets required in order that it may be possible to book a passenger from every station to every other is

(a) $\frac{10!}{8!}$ (b) $10!\,2!$ (c) $\frac{10!}{2!}$ (d) $\frac{10!}{8!2!}$

2. If the roots of $ax^2 + bx + c = 0$ are of the form $\frac{m}{m-1}$ and $\frac{m+1}{m}$ then the value of $(a + b + c)^2$ is

(a) $b^2 - 2ac$ (b) $2b^2 - ac$
(c) $b^2 - 4ac$ (d) $2(b^2 - 2ac)$

3. If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, then $(A + B)^7$ is

(a) $7(A + B)$ (b) $7 \cdot I_{3\times 3}$
(c) $64(A + B)$ (d) $128\, I_{3\times 3}$

4. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5$ is equal to

(a) $2a_1a_5$ (b) $3a_1a_5$
(c) $8a_1a_5$ (d) $4a_1a_5$

5. In a conference 10 speakers are to give their speeches one after another. Find the probability of the event if S_1 speaks before S_2 and S_2 speaks before S_3 and the remaining 7 speakers have no objection to speak at any number.

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$ (d) $\frac{3}{10}$

6. The remainder when 5^{97} is divided by 52 is

(a) 2 (b) 3 (c) 5 (d) 10

7. If $2x = -1 + \sqrt{3}i$, then the value of $(1 - x^2 + x)^6 - (1 - x + x^2)^6 =$

(a) 32 (b) -64 (c) 64 (d) 0

8. If two of the roots of the equation $\lambda - 2(x^2 + x + 1)^2 - (\lambda + 2)(x^4 + x^2 + 1) = 0$ are equal, then the value(s) of 'λ' is (are)

(a) $\{-4, 4\}$ (b) $\{-2, 2\}$
(c) $\{2\}$ (d) $\{4\}$

9. A total of 'a' litres of pure acid were drawn from a tank containing 24 litres of pure acid and were replaced by water. The result was thoroughly mixed to obtain a homogeneous solution and then 'a' litres of solution were drawn off, and again replaced by water, and again throughly mixed. This procedure was performed 3 times and the tank then contained 3 litres of pure acid. The value of 'a' is

(a) 16 (b) 8 (c) 12 (d) 4

10. The sum of the coefficients of the polynomial obtained by collection of the like terms after the expansion of $(1 - 2x + 2x^2)^{743}(2 + 3x - 4x^2)^{744}$ is

(a) 2974 (b) 1487 (c) 1 (d) 0

SECTION-II

PARAGRAPH TYPE

This section contains 2 paragraphs. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Paragraph for Q. No. 11 to 13

Let p be a prime number and n be a positive integer, then exponent of p is $n!$ which is denoted by $E_p(n!)$ and is given by

$$E_p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots\ldots\ldots+\left[\frac{n}{p^k}\right],$$

where $p^k < n < p^{k+1}$ and $[x]$ denotes the integral part of x.

If we isolate the power of each prime contained in any number N, then N can be written as

$N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots\ldots\ldots$ where $p_1, p_2, p_3, \ldots$ are prime numbers and $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots$ are natural numbers.

11. The number of zeroes at the end of 108! is

(a) 24 (b) 25 (c) 26 (d) 21

12. The number of prime numbers among the numbers,

$101! + 2, 101! + 3, 101! + 4, \ldots\ldots\ldots, 101! + 101$ is

(a) 31 (b) 29

(c) 53 (d) none of these

13. The last non-zero digit in 20! must be equal to

(a) 2 (b) 3 (c) 4 (d) 9

Paragraph for Q. No. 14 to 16

The numbers 1, 3, 6, 10, 15, 21, 28, ..., are called triangular numbers. Let t_n denotes the n^{th} triangular number then it can be observed that $t_1 = 1$, $t_2 = 3$, $t_n = t_{n-1} + n$. Answer the following questions.

14. t_{100} must be equal to

(a) 5050 (b) 5151

(c) 5252 (d) none of these

15. If m is the n^{th} triangular number, then

(a) $n=\dfrac{\sqrt{1+8m}+1}{2}$ (b) $n=\dfrac{\sqrt{1+8m}-1}{2}$

(c) $n=\dfrac{\sqrt{1+4m}-1}{2}$ (d) none of these

16. The number of positive integers lying between t_{50} and t_{51} must be

(a) 50 (b) 51

(c) 52 (d) none of these

SECTION-III

MATCHING LIST TYPE (ONLY ONE OPTION CORRECT)

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which ONLY ONE is correct.

17. Match the following:

	Column I		Column II
(P)	If n be the number of ways in which 12 different books can be distributed equally among 3 persons, then $\dfrac{(4!)^4 n}{12!}=$	1.	55
(Q)	If λ be the number of integral solutions of $x+y+z=15$ such that $x \geq 1, y \geq 2$ and $z \geq 3$, then $\lambda =$	2.	28
(R)	If λ be maximum number of points of intersection of 8 straight lines, then $\lambda =$	3.	40
(S)	A car will hold 2 persons in the front seat and 1 in the rear seat. If among 6 persons only 2 can drive, the number of ways in which car can be filled is λ, then $\lambda =$	4.	24

	P	Q	R	S
(a)	4	1	2	3
(b)	1	4	2	3
(c)	4	1	3	2
(d)	4	3	2	1

18. Match the following:

	Column I		Column II
(P)	M_r is defined as : $M_r=\begin{bmatrix} r-1 & \dfrac{1}{r} \\ 1 & \dfrac{1}{(r-1)^2} \end{bmatrix}$ and $\lvert M_r\rvert$ is the corresponding determinant value of M_r, then $\lim\limits_{n\to\infty} \lvert M_2\rvert+\lvert M_3\rvert+\ldots+\lvert M_n\rvert=$	1.	0

(Q)	If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix}$ $= \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$	2.	4
(R)	If $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ and a matrix C is defined as $C = (BAB^{-1})(B^{-1}A^TB)$, where $\|B\| \neq 0$, then $\|C\| = k^2$, $(k \in N)$ where $k =$	3.	2
(S)	If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^4 = -\lambda I$, then λ equals	4.	1

	P	Q	R	S
(a)	4	1	2	4
(b)	2	3	4	1
(c)	1	3	2	4
(d)	4	3	2	2

19. Match the following:

Column I		Column II	
(P)	Let r^{th} term of a series be given by $T_r = \frac{r}{1-3r^2+r^4}$. Then $\left(\lim_{n\to\infty} \sum_{r=1}^{n} T_r\right)+1 =$	1.	1/8
(Q)	The sum of an infinitely decreasing G.P. is equal to 4 and the sum of the cubes of its terms is equal to 64/7. Then 5^{th} term of the progression is	2.	3
(R)	If the product of three positive real numbers say a, b, c be 27, then the minimum value of $(ab + bc + ca)/81$ is equal to	3.	1/2
(S)	Let $p, q, r \in R^+$ and $27\,pqr \geq (p + q + r)^3$ and $3p + 4q + 5r = 12$ then $p^3 + q^4 + r^5$ is equal to	4.	1/3

	P	Q	R	S
(a)	4	3	2	1
(b)	3	1	2	4
(c)	1	2	3	4
(d)	3	1	4	2

20. Match the following:

Column I		Column II	
(P)	Any two small squares on a chess board are chosen at random. Probability that they have a common side is	1.	$\frac{1}{10}$
(Q)	One of ten keys opens the door. If we try the keys one after another, then the probability that the door is opened on the tenth attempt is	2.	0
(R)	The probability of obtaining no head in an infinite sequence of independent tosses of a coin is	3.	$\frac{7}{144}$
(S)	Two squares are chosen at random from the small squares drawn on a chess board. The chance that the two squares chosen have exactly one corner in common is	4.	$\frac{1}{18}$

	P	Q	R	S
(a)	3	2	1	4
(b)	4	1	2	3
(c)	1	3	2	4
(d)	2	3	1	2

ANSWERS

PAPER - I

1. (a, b, c, d) **2.** (a, b)
3. (c, d) **4.** (b) **5.** (a, b)
6. (a, c) **7.** (b, c) **8.** (a, c)
9. (a, b, d) **10.** (b, c) **11.** (5)
12. (4) **13.** (5) **14.** (1) **15.** (1) **16.** (6)
17. (9) **18.** (3) **19.** (7) **20.** (1)

PAPER - II

1. (a) **2.** (c) **3.** (c) **4.** (d) **5.** (a)
6. (c) **7.** (d) **8.** (a) **9.** (c) **10.** (c)
11. (b) **12.** (d) **13.** (c) **14.** (a) **15.** (b)
16. (a) **17.** (a) **18.** (d) **19.** (d) **20.** (b)

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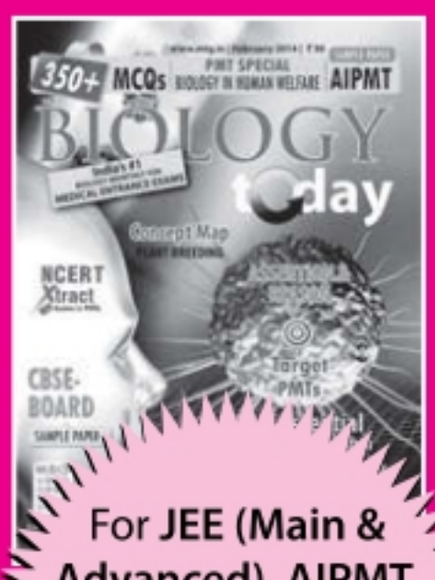

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